Optimal Beliefs in the Long Run

Yue Yuan
Washington University in St. Louis
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Abstract
People have the natural tendency to be optimistic and believe that good outcomes in the future are more likely, but also want to avoid overestimation that could result in bad decision-making. Brunnermeier and Parker (2005, 2007) established an optimal beliefs framework that balances these two incentives. This paper follows and extends the optimal beliefs framework to consider optimal beliefs in the long run after successive generations. Assuming no short-selling, result shows that in almost all cases there does not exist a stable and interior long-term optimal belief.

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1. Introduction

In standard utility theory, utility functions generally contain states and actions as arguments. Beliefs do not affect people’s sense of well-being, but rather only enter an agent’s decision making process via maximizing the expected utility function.

However, a substantial body of psychological research shows that beliefs do affect people’s sense of well-being in a very direct way. Such a literature lays a strong empirical foundation for this paper and confirms the necessity of understanding an agent’s beliefs in a new framework. More specifically, psychologists recognize that people experience emotions related to the uncertainty about the future, such as hopelessness, confidence, and anxiety. In fact, anticipatory anxiety is one of the most researched fields of psychology.

Economists have also attempted to incorporate anticipatory emotions as sources of utility. Loewenstein (1987) builds a model that assumes that a person’s instantaneous utility is the sum of utility from that period and some function of the discounted utility in future periods. This model is used to explain why one might bring forward an unpleasant experience to shorten the feeling of unhappiness (e.g., the speeding up of a dental appointment) but delay a pleasant experience to enjoy a stronger feeling of happiness (e.g., the prolonged storage of a bottle of expensive champagne). The significance of Loewenstein’s paper is that it recognizes much of our feeling of well-being arise from emotions associated with anticipation. However, Loewenstein’s model applies only to deterministic outcomes, thus does not focus on explaining an agent’s behavior driven by uncertainty about the future.

Caplin and Leahy (2001) address this issue of Loewenstein’s model. They allow time inconsistency of individual preferences in their model. They recognize that as time passes, so do anticipatory emotions, and agents’ preferences may change as a result. However, their general model of decision making under uncertainty is relatively unstructured. This lack of structure makes Caplin and Leahy (2001) model vulnerable to such criticisms that alternatives to rational expectations do not provide discipline.

In fact, since Muth (1960, 1961) and Lucas (1976), almost all research has adopted the rational expectations assumption that subjective and objective beliefs coincide. Proponents of rational expectations base their arguments mainly on two grounds. First, rationality provides discipline that is essential for modeling agents’ behaviors. Second, agents have the incentive to be rational because they are as well off as they can be holding rational expectations. The second argument, however, holds only if we assume that agents care about the future but at the same time have their anticipatory emotions about the future not affect their instantaneous well-being.

Brunnermeier and Parker (2005) view the aforementioned assumption as inconsistent. In their model, they assume that forward-looking agents care about expected future utility flow, and hence have higher instantaneous well-being if they are optimistic about the future. The optimal expectations framework established in Brunnermeier and Parker (2005) involves a two-stage decision making process. In stage 1, agent chooses “optimally” subjective beliefs subject to the optimal actions of stage 2. In stage 2, the agent solves the portfolio allocation problem given subjective beliefs. Brunnermeier and Parker (2005)’s model addresses the inconsistency in the rational expectations assumption. Moreover, it provides discipline just as the rational expectations models: biases in beliefs are determined endogenously by the economic environment.
Overall, beliefs impact instantaneous well-being directly through anticipatory emotions of the future flow utility and indirectly through their effect on portfolio allocations. This result is consistent with the abundance of psychological research.

However, the setting of the Brunnermeier and Parker (2005) model is a two-period economy. Would the results still hold in an economy with finitely many periods, given that we apply the same optimal expectations framework? This is what we want to further investigate. Our model of beliefs follows and extends the optimal expectations framework of Markus K. Brunnermeier and Jonathan A. Parker (2005). There are three main elements in their approach. Our approach extends Brunnermeier and Parker’s (2005) optimal expectations framework and has one additional key element.

**Optimal Beliefs.** First, in any given period of time \( T \), we assume that people care about current utility flow in time \( T \) and expected utility flow in time \( T + 1 \). Therefore, forward-looking agents are better off if they are optimistic about the future. For example, agents who care about expected future utility flow will be happier if they overestimate the payoff of their investments.

Second, the aforementioned optimism distorts agents’ beliefs and worsens realized outcome. Distorted beliefs would lead to suboptimal decision making, and lower level of utility on average ex post.

The third key element is a tradeoff between the first two. We define the agents’ well-being as the average of their expected present discounted value of utility flows. Therefore, agents derive subjective beliefs that maximize their well-being, thus balancing the benefits of optimism ex ante and the costs of acting on distorted beliefs ex post.

**Successive Generations.** Our model has one additional key element. We believe that if the first generation of agents, \( G_1 \), derive subjective beliefs to maximize their well-being – namely the average of their expected present discounted value of utility flows – in one particular period, subsequent generations will do so as well in every period over time. The natural question is what would be the agents’ subjective belief in the long run? The fourth key element is an iteration of subjective beliefs. Therefore, in period \( T \), generation \( G_T \) cohorts derive their \( T^{th} \) period subjective beliefs based on their \( T^{th} \) period objective probabilities. And in period \( T + 1 \), generation \( G_T \)'s children take their parents' \( T^{th} \) period subjective beliefs as their \( (T + 1)^{th} \) period objective probabilities and derive their own \( (T + 1)^{th} \) period subjective beliefs.

2. Model

We consider a world where the uncertainty can be described by 2 states where no short-selling is allowed.

2.1 Agents

The iterative scheme of our model has the flavor of an Overlapping Generations model. Suppose that \( t = 1, 2, ..., \) and that at every time \( t \) there is born a new generation \( G_t \) of individuals who live for two periods. There does not exist an initial generation \( G_0 \) at around \( t = 0 \) that lives for only 1 period. Every generation consists of a number of homogeneous agents, and assume that population growth rate is zero. Therefore, the number of agents in every generation can be normalized to 1. Agents have two-period economic lives and
do not care about future generations. Hence there are no bequests, no dynastic behavior, no altruism.

At some time period $T$, consider the generation $G_T$. In the first period ($t = T$), agents form optimal beliefs ($\hat{\pi}_{1,T}$, $\hat{\pi}_{2,T}$) and allocate their portfolios ($c_{1,T}^*$, $c_{2,T}^*$), based on the objective probabilities they perceive of the 2 states of the world ($\pi_{1,T}$, $\pi_{2,T}$). In the second period ($t = T + 1$), agents receive the realized profits of their portfolio allocation choices and die by the end of the second period.

For every generation, agents pass their optimal beliefs on to their descendants, and descendants regard their parents’ optimal beliefs as objective probabilities of the world. Mathematically, $\pi_{1,T} = \hat{\pi}_{1,T-1}$ and $\pi_{2,T} = \hat{\pi}_{2,T-1}, \forall T = 2, 3, \ldots$. In the initial period $t = 1$, the first generation $G_1$ will be given objective probabilities $\pi_{1,0}$ and $\pi_{2,0}$.

2.2 Utility Function

Assume an investor has the exponential utility function $u(c) = 1 - e^{-\alpha c}$, where $\alpha > 0$. The $\alpha$ represents the agents’ degree of risk aversion. As $\alpha$ becomes larger, the utility function displays more risk aversion. The exponential utility function has the property of modeling constant absolute risk-aversion behavior. Therefore our assumption is that the agents’ attitude towards risk is invariant with the amount of wealth they have accumulated. Therefore, in the agents’ budget constraint, $c_1 p_1 + c_2 p_2 = w$, we can conveniently normalize their wealth to 1 and re-write the budget constraint as $c_1 p_1 + c_2 p_2 = 1$, without loss of generality.

Another interesting property is that with the exponential utility function we will be able to give closed-form solutions to the proposed problem, though in this paper we do not provide all the closed-form solutions for conciseness. In Brunnermeier and Parker (2005) they use the logarithmic utility function, which does not give closed-form solutions.

However, corner solutions are possible with the exponential utility function. This is different from using the logarithmic utility function, with which corners solutions are effectively avoided by penalizing extreme beliefs with infinitely negative expected utility. Therefore, in our model, there is a chance that agents choose extreme beliefs and bet all their money in one particular state. As we will see, extreme beliefs are exactly what would happen in the long run.

2.3 Well-Being Function

Each agent’s beliefs maximize his well-being, defined as the average expected utility across period $T$ and period $T + 1$ when actions are optimal given subjective beliefs. That is, $\hat{\pi}$ maximizes $\frac{1}{2} E[V_T + V_{T+1}]$ subject to the constraints that the $\hat{\pi}_{1,T}$ and $\hat{\pi}_{2,T}$ are probabilities and that portfolio choices are optimal given $\hat{\pi}_{1,T}$ and $\hat{\pi}_{2,T}$.

This well-being function is similar to that proposed in Andrew J. Caplin and John Leahy (2000), and analogous arguments support our use of this function.

If the economy has 2 states, in order to determine the optimal beliefs, the investor only needs to maximize $\mathcal{W} = \sum_{s=1}^{2} \pi_{s,T} u(c_{s,T}^*) + \sum_{s=1}^{2} \pi_{s,T} u(c_{s,T}^*)$ with respect to $\hat{\pi}_{1,T}$ and $\hat{\pi}_{2,T}$.

With such a well-being function, beliefs impact well-being directly through anticipation of future utility and indirectly through their effect on portfolio choices.
2.4 Portfolio Choices

An agent’s optimal portfolio allocation choices, \( (c^*_1,T, c^*_2,T) \), maximize his expected utility given his subjective beliefs, \( (\hat{\pi}_1,T, \hat{\pi}_2,T) \).

Formally, \( (c^*_1,T, c^*_2,T) \) are obtained though the following:

\[
\max_{(c_1,T,c_2,T)} \left[ \hat{\pi}_{1,T}u(c_{1,T}) + \hat{\pi}_{2,T}u(c_{2,T}) \right] \text{ subject to } p_1c_{1,T} + p_2c_{2,T} = 1 \text{ and } c_{1,T}, c_{2,T} \geq 0,
\]

where \( p_1, p_2 > 0 \) are the prices of the Arrow-Debreu security yielding one unit in state 1 and 2, respectively.

2.5 Successive Generations

As discussed, for every generation, agents pass their optimal beliefs on to their descendants, and descendants take their parents’ optimal beliefs as objective probabilities of the world. In the initial period \( t = 1 \), the first generation \( G_1 \) will be given objective probabilities \( \pi_{1,0} \) and \( \pi_{2,0} \). Therefore, starting from the second generation, descendants’ objective beliefs are characterized as the following: \( \pi_{1,T} = \hat{\pi}_{1,T-1} \) and \( \pi_{2,T} = \hat{\pi}_{2,T-1}, \forall T = 2,3,... \). With objective probabilities specified, descendants will repeat generating optimal subjective beliefs and optimal portfolio choices the same way their parents did.

2.6 In a Nutshell

In an overlapping generations framework, generation \( G_T \) of cohorts choose optimal beliefs so as to maximize their well-being function:

\[
W = \sum_{s=1}^{2} \pi_{s,T}(1 - e^{-\alpha c_{s,T}(\hat{\pi}_{T})}) + \sum_{s=1}^{2} \hat{\pi}_{s,T}(1 - e^{-\alpha c_{s,T}(\hat{\pi}_{T})}),
\]

where

(a) \( c_{s,T}(\hat{\pi}_{T}) \) is obtained through maximizing the expected utility function given subjective beliefs:

\[
\max_{(c_{1,T},c_{2,T})} \left[ \hat{\pi}_{1,T}u(c_{1,T}) + \hat{\pi}_{2,T}u(c_{2,T}) \right] \text{ subject to } p_1c_{1,T} + p_2c_{2,T} = 1 \text{ and } c_{1,T}, c_{2,T} \geq 0,
\]

(b) objective probabilities are inherited from the previous generation:

\[
\pi_{1,T} = \hat{\pi}_{1,T-1} \text{ and } \pi_{2,T} = \hat{\pi}_{2,T-1}, \forall T = 2,3,...
\]

3. Results and Discussions

3.1 Optimal Portfolio Choices

**Proposition 1.** (Existence and uniqueness of optimal portfolio choices) Assuming no short-selling, at any period \( t \), the generation \( G_1 \) cohorts’ optimal portfolio choices, \( c^*_1(t) \) and \( c^*_2(t) \), exist and are unique.
Proof. At any period $t$,

$$V_t = \hat{\pi}_{1t} u(c_{1t}) + \hat{\pi}_{2t} u(c_{2t})$$

$$= \hat{\pi}_{1t} (1 - e^{-\alpha c_{1t}}) + \hat{\pi}_{2t} (1 - e^{-\alpha c_{2t}})$$

$$= \hat{\pi}_{1t} (1 - e^{-\alpha c_{1t}}) + (1 - \hat{\pi}_{1t}) (1 - e^{-\alpha \frac{1-c_{1t} p_1}{p_2}}).$$

Notice that $V_t$ is continuous in a closed and bounded set of intervals with $c_{1t} \in [0, \frac{1}{p_1}]$ and $c_{2t} \in [0, \frac{1}{p_2}]$. Therefore, by the Extreme Value Theorem, we know that $V_t$ must attain its maximum in the set of intervals. This establishes the existence of maxima.

To get the unique optimal portfolio choices, notice that the problem we need to solve is expressed in Equation (3) in the previous section. We write out the F.O.C. for the maximization problem:

$$\max_{(c_{1T}, c_{2T})} \left[ \hat{\pi}_{1T} u(c_{1T}) + \hat{\pi}_{2T} u(c_{2T}) \right] \text{ subject to } p_1 c_{1T} + p_2 c_{2T} = 1 \text{ and } c_{1T}, c_{2T} \geq 0,$$

(5)

We solve this constrained maximization problem by writing out the Lagrangian:

$$\mathcal{L} = \left[ \hat{\pi}_{1T} u(c_{1T}) + \hat{\pi}_{2T} u(c_{2T}) \right] - \lambda (p_1 c_{1T} + p_2 c_{2T} - 1)$$

(6)

$c_{1t}^*$ is solved as: $c_{1t}^* = \frac{p_2 \log (\frac{\hat{\pi}_{1t} p_2}{\alpha (p_1 + p_2)}) + \alpha}{\alpha (p_1 + p_2)}$. However, notice that $p_1 c_{1t} + p_2 c_{2t} = 1$ and $c_{1t}, c_{2t} \geq 0$. Hence $0 \leq c_{1t}^* \leq \frac{1}{p_1}$. When $\frac{p_2 \log (\frac{\hat{\pi}_{1t} p_2}{\alpha (p_1 + p_2)}) + \alpha}{\alpha (p_1 + p_2)} > \frac{1}{p_1}$, namely when $\hat{\pi}_{1t} > \frac{p_1}{p_1 + p_2 e^{-\alpha / p_1}}$, then $c_{1t}^* \approx \frac{1}{p_1}$. When $\frac{p_2 \log (\frac{\hat{\pi}_{1t} p_2}{\alpha (p_1 + p_2)}) + \alpha}{\alpha (p_1 + p_2)} < 0$, namely when $\hat{\pi}_{1t} \approx \frac{p_1}{p_1 + p_2 e^{\alpha / p_1}}$, then $c_{1t}^* = 0$.

Hence,

$$c_{1t}^* = \begin{cases} 0 & \text{ if } \hat{\pi}_{1t} \leq \frac{p_1}{p_1 + p_2 e^{\alpha / p_2}} \\ \frac{p_2 \log (\frac{\hat{\pi}_{1t} p_2}{\alpha (p_1 + p_2)}) + \alpha}{\alpha (p_1 + p_2)} & \text{ if } \frac{p_1}{p_1 + p_2 e^{\alpha / p_2}} < \hat{\pi}_{1t} < \frac{p_1}{p_1 + p_2 e^{\alpha / p_1}} \\ \frac{1}{p_1} & \text{ if } \hat{\pi}_{1t} \geq \frac{p_1}{p_1 + p_2 e^{\alpha / p_1}} \end{cases}$$

Apply the relation that $p_1 c_{1t}^* + p_2 c_{2t}^* = 1$, we also get that

$$c_{2t}^* = \begin{cases} \frac{1}{p_2} & \text{ if } \hat{\pi}_{1t} \leq \frac{p_1}{p_1 + p_2 e^{\alpha / p_2}} \\ \frac{p_1 \log (\frac{\hat{\pi}_{1t} p_1}{\alpha (p_1 + p_2)}) + \alpha}{\alpha (p_1 + p_2)} & \text{ if } \frac{p_1}{p_1 + p_2 e^{\alpha / p_2}} < \hat{\pi}_{1t} < \frac{p_1}{p_1 + p_2 e^{-\alpha / p_1}} \\ 0 & \text{ if } \hat{\pi}_{1t} \geq \frac{p_1}{p_1 + p_2 e^{-\alpha / p_1}} \end{cases}$$
Hence completes the proof.

Therefore, for any given \( p_1, p_2, \) and \( \alpha, \) there always exist two points \( 0 < \frac{p_1}{p_1 + p_2 e^{\alpha / p_1}} < \frac{p_1}{p_1 + p_2 e^{\alpha / p_2}} < 1 \) such that if the distorted belief \( \hat{\pi}_{1t} \in [0, \frac{p_1}{p_1 + p_2 e^{\alpha / p_1}}], \) the agent would want to bet all his money in state 2. Whereas if the distorted belief \( \hat{\pi}_{1t} \in [\frac{p_1}{p_1 + p_2 e^{\alpha / p_2}}, 1], \) the agent would want to bet all his money in state 1. This result is intuitive, because the larger \( \hat{\pi}_{1t} \) becomes, the more optimistic the agent is about state 1. When the agent gets too optimistic to a point in which the benefits of optimism ex ante completely overwhelms the costs of bad outcome by acting on distorted beliefs, the agent would be willing to put all his money in state 1. In this case, the agent does not care how much she may lose ex post because she is simply satisfied with believing in state 1 no matter what. This happens in the real world, as demonstrated in the case of people purchasing lottery tickets. Most people buy lottery tickets because the possibility of winning a lot of money makes them so that they are willing to forego a few dollars even if there is a big chance that they lose those few dollars.

3.2 Optimal Subjective Beliefs

**Proposition 2.** (Existence of optimal beliefs) At any period \( t, \) the generation \( G_t \) cohorts’ optimal portfolio beliefs, \( \hat{\pi}_{1t}^* \) and \( \hat{\pi}_{2t}^*, \) exist and, in almost all cases, are unique.

In order to break down the proof of this proposition, we need several lemmas. Notice that because optimal portfolio choices \( c_{1t}^* \) and \( c_{2t}^* \) are piecewise functions with three pieces, the well-being function will need to be expressed as a piecewise function as well.

**Lemma 1.** The well-being function can be expressed as a piecewise function with three pieces.

**Proof.** By Equation (2), we have that

\[
W = (\hat{\pi}_{1,T} + \pi_{1,T})(1 - e^{-\alpha c_{1,T}^*}) + (\hat{\pi}_{2,T} + \pi_{2,T})(1 - e^{-\alpha c_{2,T}^*}),
\]

(7)

By Proposition 1, we also have that

\[
c_{1t}^* = \begin{cases} 
0 & \text{if } \hat{\pi}_{1t} \leq \frac{p_1}{p_1 + p_2 e^{\alpha / p_2}} \\
\frac{p_2 \log \left( \frac{\hat{\pi}_{1t}^p}{p_1 + p_2 e^{\alpha / p_2}} \right) + \alpha}{\alpha (p_1 + p_2)} & \text{if } \frac{p_1}{p_1 + p_2 e^{\alpha / p_2}} < \hat{\pi}_{1t} < \frac{p_1}{p_1 + p_2 e^{\alpha / p_1}} \\
\frac{1}{p_1} & \text{if } \hat{\pi}_{1t} \geq \frac{p_1}{p_1 + p_2 e^{\alpha / p_1}}
\end{cases}
\]

and
To simplify expressions, we rewrite the second pieces of $c_{1t}^*$ and $c_{2t}^*$:

$$
c_{1t}^* = \frac{p_2 \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + \alpha}{\alpha(p_1 + p_2)}
= \frac{p_2}{\alpha(p_1 + p_2)} \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + \frac{p_2 \log \left( \frac{\hat{\pi}_1}{p_1} \right) + \alpha}{\alpha(p_1 + p_2)}
= -\frac{1}{\alpha} \left[ \frac{-p_2}{p_1 + p_2} \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + \frac{-p_2 \log \left( \frac{\hat{\pi}_1}{p_1} \right) - \alpha}{p_1 + p_2} \right]
= -\frac{1}{\alpha} \left[ A \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + B \right], \text{where } A = \frac{-p_2}{p_1 + p_2} \text{ and } B = \frac{-p_2 \log \left( \frac{\hat{\pi}_1}{p_1} \right) - \alpha}{p_1 + p_2},
$$

also:

$$
c_{2t}^* = \frac{p_1 \log \left( \frac{1-\hat{\pi}_1}{\hat{\pi}_1} \right) + \alpha}{\alpha(p_1 + p_2)}
= \frac{-p_1 \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) - p_1 \log \left( \frac{\hat{\pi}_1}{p_1} \right) + \alpha}{\alpha(p_1 + p_2)}
= \frac{-p_1}{\alpha(p_1 + p_2)} \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + \frac{-p_1 \log \left( \frac{\hat{\pi}_1}{p_1} \right) + \alpha}{\alpha(p_1 + p_2)}
= -\frac{1}{\alpha} \left[ \frac{-p_1}{p_1 + p_2} \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + \frac{-p_1 \log \left( \frac{\hat{\pi}_1}{p_1} \right) - \alpha}{p_1 + p_2} \right]
= -\frac{1}{\alpha} \left[ C \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + D \right], \text{where } C = \frac{p_1}{p_1 + p_2} \text{ and } D = \frac{p_1 \log \left( \frac{\hat{\pi}_1}{p_1} \right) - \alpha}{p_1 + p_2}.
$$

Therefore,

$$
W_\epsilon = (\hat{\pi}_1 + \pi_1)[1 - e^{-\alpha c_1^*}] + (\hat{\pi}_2 + \pi_2)[1 - e^{-\alpha c_2^*}]
= (\hat{\pi}_1 + \pi_1)[1 - e^{A \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + B}] + (2 - \hat{\pi}_1 - \pi_1)[1 - e^{C \log \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) + D}]
= (\hat{\pi}_1 + \pi_1)[1 - e^{B \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) A} - (\hat{\pi}_1 + \pi_1)[1 - e^{D \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) C}] + 2[1 - e^{D \left( \frac{\hat{\pi}_1}{1-\hat{\pi}_1} \right) C}]$$
Putting everything together,

$$\mathcal{W} = \begin{cases} 
(\hat{\pi}_1 + \pi_1)(1 - e^{-\alpha/p_1}), \hat{\pi}_1 \in \left[\frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}, 1\right] \\
(\hat{\pi}_1 + \pi_1)[1 - e^B(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^A] + (2 - \hat{\pi}_1 - \pi_1)[1 - e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^C], \hat{\pi}_1 \in \left(\frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}, \frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}\right) \\
(2 - \hat{\pi}_1 - \pi_1)(1 - e^{-\alpha/p_2}), \hat{\pi}_1 \in [0, \frac{p_1}{p_1 + p_2 e^{-\alpha/p_2}}]
\end{cases}$$

Given the well-being function being piecewise, we find the global optimal belief in three steps. Notice that the well-being function has three pieces. The subscripts of $W^*$ indicates the maximum well-being in a particular subinterval. We label the optimal beliefs in every piece as $W^*_1$, $W^*_2$, and $W^*_3$, corresponding to the subinterval $\hat{\pi}_1 \in [0, \frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}]$, $\left(\frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}, \frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}\right)$, and $\left[\frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}, 1\right]$, respectively. In the first step, we find $W^*_1$ and $W^*_3$, which are similarly solved for. In the second step, we solve for $W^*_2$. Finally, we compare $W^*_1$, $W^*_2$, and $W^*_3$. The global optimal belief is the distorted belief associated with the maximum of the three.

First, we find $W^*_1$ and $W^*_3$.

**Lemma 2.** $W^*_1 = (2 - \pi_1)(1 - e^{-\alpha/p_2})$, and $W^*_3 = (1 + \pi_1)(1 - e^{-\alpha/p_1})$.

**Proof.** The two extreme cases are relatively easy to solve. First, consider the case when $\hat{\pi}_1 \in [0, \frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}]$. In this case, $c^*_{1t} = 0$ and $c^*_{1t} = 1/p_2$. Therefore, the agent’s well-being function is:

$$W_\infty = (2 - \hat{\pi}_1 - \pi_1)(1 - e^{-\alpha/p_2}).$$  

To maximize this well-being function, we only need to take $\hat{\pi}_1 = 0$, and the maximum well-being in this segment of the piece function is $W^*_1 = (2 - \pi_1)(1 - e^{-\alpha/p_2})$.

Second, consider the case when $\hat{\pi}_1 \in \left(\frac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}, 1\right]$. In this case, $c^*_{1t} = 1/p_1$ and $c^*_{1t} = 0$. Therefore, the agent’s well-being function is:

$$W_3 = (\hat{\pi}_1 + \pi_1)(1 - e^{-\alpha/p_1}).$$

To maximize this well-being function, we only need to take $\hat{\pi}_1 = 1$, and the maximum well-being in this segment of the piece function is $W^*_3 = (1 + \pi_1)(1 - e^{-\alpha/p_1})$. 

To find $W^*_2$, we need two additional lemmas.

**Lemma 3.** The first derivative of $W^*_2$ is $\frac{\partial W}{\partial \pi_1} = \left(\frac{\hat{\pi}_1}{\pi_1(1 - \hat{\pi}_1)^A}\right) \cdot [- (1 + p)^2 \hat{\pi}_1^2 + (1 + p)(2 + p)\hat{\pi}_1^2 - (1 + 2p)\hat{\pi}_1 + p\pi_1]$, where $p = p_2/p_1$ denotes the price ratio.
Proof.

\[ W_\varepsilon = (\hat{\pi}_1 + \pi_1)[1 - e^B(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^A] - (\hat{\pi}_1 + \pi_1)[1 - e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^C] + 2[1 - e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^C] \]

\[ = (\hat{\pi}_1 + \pi_1)[e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^C - e^B(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^A] + 2[1 - e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^C] \]

If we define \( f(\hat{\pi}_1) = \frac{\hat{\pi}_1}{1 - \hat{\pi}_1} \), then \( \frac{\partial f(\hat{\pi}_1)}{\partial \hat{\pi}_1} = \frac{1}{(1 - \hat{\pi}_1)^2} \). Therefore,

\[ \frac{\partial W}{\partial \hat{\pi}_1} = 1 \cdot [e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^C - e^B(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^A] \]

\[ + (\hat{\pi}_1 + \pi_1)[e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-1} - e^B \cdot A \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{A-1}] \cdot \frac{1}{(1 - \hat{\pi}_1)^2} \]

\[ + 2[-e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-1}] \cdot \frac{1}{(1 - \hat{\pi}_1)^2} \]

\[ = e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^C - e^B(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^A - 2e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^C \cdot \frac{1}{\hat{\pi}_1(1 - \hat{\pi}_1)} \]

\[ + \frac{\hat{\pi}_1 + \pi_1}{\hat{\pi}_1(1 - \hat{\pi}_1)}[e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-A} - e^B \cdot A] \]

Notice that \( A = \frac{p_2}{\rho_1 + p_2} \) and \( C = \frac{\rho_1}{\rho_1 + p_2} \), so:

\[ \frac{\partial W}{\partial \hat{\pi}_1} = e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-A} - 2e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-A} \cdot \frac{1}{\hat{\pi}_1(1 - \hat{\pi}_1)} \]

\[ + \frac{\hat{\pi}_1 + \pi_1}{\hat{\pi}_1(1 - \hat{\pi}_1)}[e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-A} - e^B \cdot A] \]

Since \( C - A = 1 \), we have:

\[ \frac{\partial W}{\partial \hat{\pi}_1} = e^D(\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-A} - 2e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-A} \cdot \frac{1}{\hat{\pi}_1(1 - \hat{\pi}_1)} \]

\[ + \frac{\hat{\pi}_1 + \pi_1}{\hat{\pi}_1(1 - \hat{\pi}_1)}[e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1 - \hat{\pi}_1})^{C-A} - e^B \cdot A] \]
To simplify, multiply both sides by \( \hat{\pi}_1(1-\hat{\pi}_1)^2 \), and we get a 3-degree polynomial with respect to \( \hat{\pi}_1 \):

\[
\frac{\partial W}{\partial \hat{\pi}_1}/\left[ \frac{(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})^A}{\hat{\pi}_1(1-\hat{\pi}_1)^2} \right] = e^D \hat{\pi}_1^2 (1-\hat{\pi}_1) - e^B \hat{\pi}_1 (1-\hat{\pi}_1)^2 - 2e^D C \hat{\pi}_1 \\
+ (\hat{\pi}_1 + \pi_1) e^D - (\hat{\pi}_1 + \pi_1)(1-\hat{\pi}_1) e^B A
\]

Rearranging the equation gives us:

\[
\frac{\partial W}{\partial \hat{\pi}_1}/\left[ \frac{(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})^A}{\hat{\pi}_1(1-\hat{\pi}_1)^2} \right] = [-e^D - e^B C e^D - (1 + A(1-\pi_1))] \hat{\pi}_1 - A \hat{\pi}_1
\]

Notice further that \( B = \frac{-p_2 \log(\frac{p_2}{p_1}) - \alpha}{p_1 + p_2} \) and \( D = \frac{p_1 \log(\frac{p_2}{p_1}) - \alpha}{p_1 + p_2} \).

Therefore, \( D - B = \log(\frac{p_1}{p_2}) \). Apply this result, and we have:

\[
\frac{\partial W}{\partial \hat{\pi}_1}/\left[ \frac{(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})^A e^B}{\hat{\pi}_1(1-\hat{\pi}_1)^2(1+p)} \right] = (1 + p)^2 \hat{\pi}_1^3 - (1 + p)(2 + p) \hat{\pi}_1^2 + (1 + 2p) \hat{\pi}_1 - p \pi_1,
\]

assuming that \( p = p_2 / p_1 \) denotes the price ratio. Hence,\( \frac{\partial W}{\partial \hat{\pi}_1} = \left[ \frac{(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})^A e^B}{\hat{\pi}_1(1-\hat{\pi}_1)^2(1+p)} \right] \cdot [- (1 + p)^2 \hat{\pi}_1^3 + (1 + p)(2 + p) \hat{\pi}_1^2 - (1 + 2p) \hat{\pi}_1 + p \pi_1] \) \( (10) \)

\textbf{Corollary 1.} Notice that \( \frac{\hat{\pi}_1}{(1-\hat{\pi}_1)^2(1+p)} \) is always a positive number. Therefore, in order to determine the sign (positive/negative) of the first derivative, \( \frac{\partial W}{\partial \hat{\pi}_1} \), we only need to check the sign of

\[
-(1 + p)^2 \hat{\pi}_1^3 + (1 + p)(2 + p) \hat{\pi}_1^2 - (1 + 2p) \hat{\pi}_1 + p \pi_1
\]

Expression (11) is essentially a transformation of the 1st order derivative of the second segment of the well-being function. Therefore, we can solve for the roots of the third degree polynomial (11) in the interval \( \hat{\pi}_1 \in (\frac{p_1}{p_1 + p_2 e^\alpha p_2}, \frac{p_1}{p_1 + p_2 e^{-\alpha} p_1}) \), and check which root, when substituted back into the well-being function, gives the maximum well-being \( W_2^* \).

\textbf{Lemma 4.} The second derivative of \( W_2 \) is \( \frac{\partial^2 W}{\partial \hat{\pi}_1^2} = \frac{(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})^A e^B p}{(1-\pi_1)^2(1+p)^2} [-3(1 + p) \hat{\pi}_1^2 + (1 + 2p + 3(1 + p) \pi_1) \hat{\pi}_1 - (1 + 2p) \hat{\pi}_1] \).
Proof. Since

\[
\frac{\partial W}{\partial \hat{\pi}_1} = e^D(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})C - e^B(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})A - 2e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1-\hat{\pi}_1})C \cdot \frac{1}{\hat{\pi}_1(1-\hat{\pi}_1)} + \frac{\hat{\pi}_1 + \pi_1}{\hat{\pi}_1(1-\hat{\pi}_1)} [e^D \cdot C \cdot (\frac{\hat{\pi}_1}{1-\hat{\pi}_1})C - e^B \cdot A \cdot (\frac{\hat{\pi}_1}{1-\hat{\pi}_1})A],
\]

we have the second derivative:

\[
\frac{\partial^2 W}{\partial \hat{\pi}_1^2} = e^D C(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})C^{-1} \frac{1}{(1-\hat{\pi}_1)^2} - e^B A(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})A^{-1} \frac{1}{(1-\hat{\pi}_1)^2} - 2e^D C(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})C^{-1} \frac{1}{\hat{\pi}_1(1-\hat{\pi}_1)^2} + \frac{\hat{\pi}_1 + \pi_1}{\hat{\pi}_1(1-\hat{\pi}_1)} [e^D C(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})C - e^B A(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})A] + \frac{\hat{\pi}_1 + \pi_1}{\hat{\pi}_1(1-\hat{\pi}_1)} [e^D C^2(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})C^{-1} \frac{1}{(1-\hat{\pi}_1)^2} - e^B A^2(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})A^{-1} \frac{1}{(1-\hat{\pi}_1)^2}]
\]

By using the fact established in Lemma 3 that \( C - A = 1 \) and \( D - B = \log \left( \frac{p_2}{p_1} \right) \), we have the following relations:

\[
\frac{\partial^2 W}{\partial \hat{\pi}_1^2} = \left(\frac{\hat{\pi}_1}{1-\hat{\pi}_1}\right)^A e^{Bp} \frac{A}{(1-\hat{\pi}_1)^3 \pi_1^2 (1+p)^2} \left[-3(1+p)\hat{\pi}_1^2 + (1 + 2p + 3(1+p)\pi_1)\hat{\pi}_1 - (1 + 2p)\hat{\pi}_1\right]
\]

(12)

\[\square\]

Corollary 2. Notice that \( \left(\frac{\hat{\pi}_1}{1-\hat{\pi}_1}\right)^A e^{Bp} \) is always a positive number. Therefore, in order to determine the sign (positive/negative) of the second derivative, \( \frac{\partial^2 W}{\partial \hat{\pi}_1^2} \), we only need to check the sign of

\[
-3(1+p)\hat{\pi}_1^2 + (1 + 2p + 3(1+p)\pi_1)\hat{\pi}_1 - (1 + 2p)\pi_1
\]

(13)

Put the above two lemmas together, and we can find \( W_2^* \):

Lemma 5. \( W_2^* \) exists and can be solved analytically.

The existence of \( W_2^* \) is established by the the function \( W_2 \)'s continuity and the compactness of its interval. To solve for \( W_2^* \) analytically, we need to find the roots of the first derivative of \( W_2 \). Then we apply the second derivative test to find those local maxima in the range \( \hat{\pi}_{1t} \in (\frac{\pi_1}{p_1+p_2e^{\alpha/p_2}}, \frac{\pi_1}{p_1+p_2e^{-\alpha/p_1}}) \). For those local maxima in the range, we find the one that gives the largest \( W_2 \) and denotes the well-being associated with this subjective belief as \( W_2^* \).

Notice that in the above proof, we did not explicitly calculate for \( W_2^* \). The reason is that the calculation will be too tedious: because equation (11) is a third degree polynomial, it could have either 1, or 2, or 3 real roots. Meticulously studying for all different cases does not provide us with much additional insight.
After obtaining $W_1^*, W_2^*$, and $W_3^*$ by using Lemma 2 and Lemma 5, we can compare the three maximum well-beings in each segment and find the global maximum well-being. The optimal subjective belief is the distorted belief associated with the global maximum well-being.

We state a Corollary without proof.

**Corollary 3.** Given fixed $p_1$, $p_2$ and $\pi_1$, as we increase $\alpha$, the global optimal belief is more likely to be interior. In other words, there exists $\alpha_0$ such that the subjective belief associated with $W_2^*$ is the optimal belief the agents adopt, for all $\alpha > \alpha_0$.

To see why this is true, here is a brief explanation:

By Lemma 2, we have:

$$W_3^* = (1 + \pi_1)(1 - e^{-\alpha/p_1}), \quad W_1^* = (2 - \pi_1)(1 - e^{-\alpha/p_2}).$$

They are the max that can be archived in the two pieces. For the second piece, we can rewrite the well-being function as:

$$W = (\hat{\pi}_1 + \pi_1)[e^D(\frac{\hat{\pi}_1}{1 - \pi_1})^C - e^B(\frac{\hat{\pi}_1}{1 - \pi_1})^A] + 2[1 - e^D(\frac{\hat{\pi}_1}{1 - \pi_1})^C]$$

Only $e^B$ is dependent on $\alpha$, and $B = -\frac{p_2 \log (\frac{p_2}{p_1}) - \alpha}{p_1 + p_2}$. Therefore as $\alpha$ increases, every point in the second piece gets closer to 2. Eventually, the second piece can get infinitesimally closer to 2. Whereas for $W_1^*$ and $W_3^*$ the largest they can be are $1 + \pi_1$ and $2 - \pi_1$, which are both smaller than 2.

A number of graphs can be helpful in gaining intuition about what happens. Figure 1 shows a several different scenarios and the corresponding optimal subjective belief the agent will adopt in the period. Notice that the dashed line is the first derivative of the second piece of the well-being function, the solid black line is the well-being function in $\hat{\pi}_1 \in (0, 1)$, and the dotted black lines are the extensions of the segment of the well-being function into the intervals $\hat{\pi}_1 \in [0, \frac{p_1}{p_1 + p_2 e^{-\alpha/p_2}}]$, and $\hat{\pi}_1 \in [\frac{p_1}{p_1 + p_2 e^{\alpha/p_2}}, 1]$. In Figure 1(a), there is only one local maximum in the second segment, and this $W_2^*$ is the highest. Therefore, the agent’s optimal subjective belief will be in the second segment, and the segment will not choose to go to the extreme by betting all his money in one state. In Figure 1(b), there is still only one local maximum in the second segment, but this time $W_1^*$ is the highest. Therefore, the agent will choose her optimal subjective belief such that all her money goes into state 2. Notice that we only changed $\alpha$ to be smaller but kept all other parameters constant. This corresponds to what we get from Corollary 4. In Figure 1(c), there are three local optima in the second segment.

### 3.3 Long Run Optimal Subjective Beliefs

After the agents in generation $G_T$ set their optimal subjective beliefs in period $T$, their descendants, generation $G_{T+1}$, are born at the beginning of period $T + 1$. For agents in generation $G_{T+1}$, they take the optimal subjective beliefs set by generation $G_T$ in period $T$ as their objective probabilities of the states of
(a) Only one local maximum in the second segment. And global maximum lies in the second piece.

(b) Only one local maximum in the second segment. And global maximum is a corner solution.

(c) Three local optima in the second segment. And global maximum is a corner solution.

Figure 1: Well-being function in different scenarios
world. They will then form their own distorted beliefs, maximize their well-being function, and decide on their own optimal subjective beliefs. This iteration keeps on and there will be some type of the long-term optimal subjective beliefs.

**Proposition 3.** There does not exist a stable and interior long term optimal belief, $\hat{\pi}^*$. 

**Proof.** Suppose we have an interior long run optimal subjective belief, say $\hat{\pi}^*$. Then, because descendants inherit objective probabilities from their parents as in Equation (4), we know that $\hat{\pi}^* = \tilde{\pi}_1 = \pi_1$ in the long run. Apply this to Equation (11), which can be rewritten as

$$-(1+p)(\tilde{\pi}^*)^3 + (1+p)(2+p)(\tilde{\pi}^*)^2 - (1+2p)\tilde{\pi}^* + p\tilde{\pi}^*. \tag{14}$$

The roots to this degree 3 polynomial are $\hat{\pi}^* = 0$, $\hat{\pi}^* = 1/(1+p)$, and $\hat{\pi}^* = 1$. Since $\hat{\pi}^*$ is interior, the only feasible candidate for stable and interior long run optimal belief is $\hat{\pi}^* = 1/(1+p)$.

Use the second derivative to check if $\hat{\pi}^* = 1/(1+p)$ is indeed a maximum. We check the sign of the second derivative as in Equation (13). The roots to Equation (13) are $\tilde{\pi}_1 = \pi_1$ and $\tilde{\pi}_1 = 1/(1+2p)$, $\pi_1 = 1/(1+p)$. There are two cases:

1. If $p = 1$, then $1/(1+p) = 1/(1+p)$, and the second derivative is always non-positive. Hence $\hat{\pi}^* = 1/(1+p)$ is indeed a local maximum.

2. If $p \neq 1$, then $1/(1+p) \neq 1/(1+2p)$. Since the second derivative at $\hat{\pi}^* = 1/(1+p)$ is 0 and the third derivative at $\hat{\pi}^* = 1/(1+p)$ is nonzero, we know that $\hat{\pi}^* = 1/(1+p)$ is an inflection point and cannot be a local maximum.

Hence, unless $p = 1$ and $\hat{\pi}^* = 1/2$, in all other cases there does not exist a stable and interior long term optimal belief. \(\square\)

4. **Numerical Examples**

Let’s work through several concrete examples for illustrative purposes.

**Example 1.** We consider a simplistic world where the uncertainty can be described by 2 states. The prices of the Arrow-Debreu security yielding one unit in each state is $p_1 = 1$ and $p_2 = 2$, so the price ratio is $p = 2$. These prices remain constant across all generations.

For agents across all generations, we assume that they have the same degree of risk aversion, with $\alpha = 5$. In addition, the first generation of cohorts perceive the objective probabilities of the states as $\pi_{1,t=1} = 1/3$ and $\pi_{2,t=1} = 2/3$.

**($t=1$).** In period 1, the first generation, $G_1$ of cohorts will go through the following decision-making procedures:

By Lemma 2, $W_{1}^* = (2 - \pi_1)(1 - e^{-\alpha/p_2}) = 5/3(1 - e^{-5/2})\forall \tilde{\pi}_{1t} \in [0, \frac{1}{1+2e^{-5/2}}]$, and $W_3^* = 4/3(1 - e^{-5})\forall \tilde{\pi}_{1t} \in [\frac{1}{1+2e^{-5}}, 1]$. 

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To solve for $\mathcal{W}_2^*$, we calculate roots of expression (8) to seek candidates for interior optima. With information provided, expression (8) is written as $\left[ -9\hat{\pi}_1^3 + 12\hat{\pi}_1^2 - 5\hat{\pi}_1 + \frac{2}{3} \right]$. Roots are $\hat{\pi}_1 = \frac{1}{3}$ and $\hat{\pi}_1 = \frac{2}{3}$. By second derivative test, we find that $\hat{\pi}_1 = \frac{1}{3}$ is an inflection point and $\hat{\pi}_1 = \frac{2}{3}$ is a local interior maximum. Therefore, the optimal subjective belief in the subinterval $\hat{\pi}_{1t} \in (\frac{1}{1+2e^{\gamma\frac{72}{\kappa}}, \frac{1}{1+2e^{\gamma\frac{72}{\kappa}}})$ is $\hat{\pi}_1 = \frac{2}{3}$.

Substituting $\hat{\pi}_1 = \frac{2}{3}$ into $\mathcal{W}_2^*$, and we found that $\mathcal{W}_2^*$ is larger than both $\mathcal{W}_1^*$ and $\mathcal{W}_3^*$. Therefore, the optimal beliefs that generation $G_2$ will adopt is $\hat{\pi}_1 = \frac{2}{3}$ and $\hat{\pi}_2 = \frac{1}{3}$. It is illustrated in Figure 2(a).

(*t=2*). In period 2, the second generation, $G_2$, of cohorts take generation 1 cohorts’ optimal beliefs as their objective probabilities. So $\pi_{1,t=2} = \frac{2}{3}$ and $\pi_{2,t=2} = \frac{1}{3}$. They will go through the following decision-making procedures:

By Lemma 2, $\mathcal{W}_1^* = (2 - \pi_1)(1 - e^{-\alpha/p_2}) = \frac{4}{3}(1 - e^{-5/2})\forall \pi_{1t} \in [0, \frac{1}{1+2e^{\gamma\frac{72}{\kappa}}}]$, and $\mathcal{W}_3^* = \frac{5}{3}(1 - e^{-5})\forall \pi_{1t} \in [\frac{1}{1+2e^{\gamma\frac{72}{\kappa}}}, 1]$.

To solve for $\mathcal{W}_2^*$, we calculate roots of expression (8) to seek candidates for interior optima. With information provided, expression (8) is written as $\left[ -9\hat{\pi}_1^3 + 12\hat{\pi}_1^2 - 5\hat{\pi}_1 + \frac{4}{3} \right]$. The only real roots is $\hat{\pi}_1 = \frac{1}{3}$ and $\hat{\pi}_1 = \frac{2}{3}$. By second derivative test, we find that $\hat{\pi}_1 = 0.8985$ is a local interior maximum. Therefore, the optimal subjective belief in the subinterval $\hat{\pi}_{1t} \in (\frac{1}{1+2e^{\gamma\frac{72}{\kappa}}, \frac{1}{1+2e^{\gamma\frac{72}{\kappa}}})$ is $\hat{\pi}_1 = 0.8985$.

Substituting $\hat{\pi}_1 = 0.8985$ into $\mathcal{W}_2^*$, and we found that $\mathcal{W}_2^*$ is larger than both $\mathcal{W}_1^*$ and $\mathcal{W}_3^*$. Therefore, the optimal beliefs that generation $G_1$ will adopt is $\hat{\pi}_1 = 0.8985$ and $\hat{\pi}_2 = 0.1015$. It is illustrated in Figure 2(b).

(*t=3*). In period 3, the third generation, $G_3$, of cohorts take generation 1 cohorts’ optimal beliefs as their objective probabilities. So $\pi_{1,t=3} = 0.8985$ and $\pi_{2,t=3} = 0.1015$. They will go through the following decision-making procedures:

By Lemma 2, $\mathcal{W}_1^* = (2 - \pi_1)(1 - e^{-\alpha/p_2}) = 1.0105(1 - e^{-5/2})\forall \pi_{1t} \in [0, \frac{1}{1+2e^{\gamma\frac{72}{\kappa}}}]$, and $\mathcal{W}_3^* = 1.8985(1 - e^{-5})\forall \pi_{1t} \in [\frac{1}{1+2e^{\gamma\frac{72}{\kappa}}}, 1]$.

To solve for $\mathcal{W}_2^*$, we calculate roots of expression (8) to seek candidates for interior optima. With information provided, expression (8) is written as $\left[ -9\hat{\pi}_1^3 + 12\hat{\pi}_1^2 - 5\hat{\pi}_1 + 1.7970 \right]$. The only real roots is $\hat{\pi}_1 \approx 0.973311$. By second derivative test, we find that $\hat{\pi}_1 = 0.973311$ is a local interior maximum. Therefore, the optimal subjective belief in the subinterval $\hat{\pi}_{1t} \in (\frac{1}{1+2e^{\gamma\frac{72}{\kappa}}, \frac{1}{1+2e^{\gamma\frac{72}{\kappa}}})$ is $\hat{\pi}_1 = 0.973311$.

However, this time, substituting $\hat{\pi}_1 = 0.8985$ into $\mathcal{W}_2^*$, and we found that $\mathcal{W}_2^*$ is larger than both $\mathcal{W}_1^*$ and $\mathcal{W}_3^*$. Therefore, the optimal beliefs that generation $G_1$ will adopt is $\hat{\pi}_1 = 1$ and $\hat{\pi}_2 = 0$. It is illustrated in Figure 2(c).

(*t=4,5,...*). $\hat{\pi}_1 = 1$ and $\hat{\pi}_2 = 0$ will remain in later periods.

Hence, overall, we can plot $\hat{\pi}_1$ in each period in Figure 3.
Figure 2: Well-being function and optimal beliefs for Example 1 in the first three periods.
Let's look at another example in which optimal beliefs remain interior in the long run.

Example 2. We consider a simplistic world where the uncertainty can be described by 2 states. The prices of the Arrow-Debreu security yielding one unit in each state is \( p_1 = 1 \) and \( p_2 = 1 \), so the price ratio is \( p = 1 \). These prices remain constant across all generations.

For agents across all generations, we assume that they have the same degree of risk aversion, with \( \alpha = 5 \). In addition, the first generation of cohorts perceive the objective probabilities of the states as \( \pi_{1,t=1} = \frac{1}{2} \) and \( \pi_{2,t=1} = \frac{1}{2} \).

\((t=1)\). In period 1, the first generation, \( G_1 \) of cohorts will go through the following decision-making procedures:

By Lemma 2, \( W^*_1 = (2 - \pi_1)(1 - e^{-\alpha/p2}) = \frac{3}{2}(1 - e^{-5})\forall \hat{\pi}_{1t} \in [0, \frac{1}{1+e^5}] \), and \( W^*_3 = \frac{3}{2}(1 - e^{-5})\forall \hat{\pi}_{1t} \in [\frac{1}{1+e^5}, 1] \).

To solve for \( W^*_2 \), we calculate roots of expression (8) to seek candidates for interior optima. With information provided, expression (8) is written as \( -4\hat{\pi}^3_1 + 6\hat{\pi}^2_1 - 3\hat{\pi}_1 + \frac{1}{2} \). The root to the polynomial is \( \hat{\pi}_1 = \frac{1}{2} \). By second derivative test, we find that \( \hat{\pi}_1 = \frac{1}{2} \) is a local interior maximum. Therefore, the optimal subjective belief in the subinterval \( \hat{\pi}_{1t} \in (\frac{1}{1+e^5}, \frac{1}{1+e^5}) \) is \( \hat{\pi}_1 = \frac{1}{2} \).

Substituting \( \hat{\pi}_1 = \frac{2}{3} \) into \( W^*_3 \), and we found that \( W^*_3 \) is larger than both \( W^*_1 \) and \( W^*_3 \). Therefore, the optimal beliefs that generation \( G_1 \) will adopt is \( \hat{\pi}_1 = \frac{1}{2} \) and \( \hat{\pi}_2 = \frac{1}{2} \). It is illustrated in Figure 4.

\( T=2, 3, \ldots \). In subsequent generations, agents are repeating the same procedures and will adopt the same optimal beliefs. The time series is depicted in Figure 5.
Figure 4: Well-being function and optimal beliefs for Example 2. Notice that all generations have the same well-being function and optimal beliefs.

Figure 5: Evolution of optimal belief $\hat{\pi}_t$ for generation $G_t$ for Example 2.
5. Concluding Remarks

This paper follows the optimal expectations framework established by Brunnermeier and Parker (2005, 2007) but takes a long-term perspective when investigating optimal beliefs. In this model, agents continue to balance between being optimistic about the future and optimizing expected realized utility flows. However, my model extends the optimal expectations framework and assumes that each generation of agents adopt their parents’ optimal beliefs as objective probabilities of the states of the world. This additional element is particularly useful in accommodating cases when states of the world do not have true probabilities or the true probabilities are not observable.

Assuming that no short-selling is allowed, the result shows that: (1), at any particular time period \( t \), the generation \( G_t \) cohorts always have unique optimal portfolio choices as well unique optimal beliefs; and (2), in the long run, except for a special case in which the two states of the world are identical and that agents start with unbiased objective beliefs about the two states, in all other cases there does not exist a stable and interior long run optimal belief. Depending on the price ratio, the agents in sufficiently late generations will always make extreme choices of betting all their wealth into one state.

This study makes the assumption that individuals only learn from their parents, and they do not observe the true probabilities of the states of the world nor learn from their grandparents or elder generations. This is useful analytical assumption for our purposes, but assuming learning from grandparents or even elder generations would be particularly helpful for a more thorough understanding. Preliminary results indicate that if individuals are able to learn from not only their parents but also their grandparents or elder generations, the rate of divergence to extreme optimal beliefs will be slower.

Future research could also check if these results are robust to different utility functions. For example, Brunnermeier and Parker (2007) assumed logarithmic utility function in their model. It is likely that long run optimal belief will converge to extreme but never reach extreme under the logarithmic utility function assumption. In addition, future work could also consider evaluating exactly how agents decide which state of the world to put all their wealth into depending on various variables such as the price ratio.
References


