Monotonic $\epsilon$-equilibria in strongly symmetric games

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Abstract

$\epsilon$-equilibrium allows for worse actions to be played with higher probability than better actions. I introduce a refinement that addresses this shortcoming: an $\epsilon$-equilibrium is monotonic if each player assigns (weakly) higher probabilities to better actions. This concept is logically independent of strong $\epsilon$-equilibrium. I study these refinements of $\epsilon$-equilibrium in games with countably many players and finitely many pure strategies, and prove the existence of monotonic $\epsilon$-equilibrium in a special class of such games—strongly symmetric games. The proof is constructive and it implies, in particular, the existence of symmetric $\epsilon$-equilibrium in strongly symmetric games. This result fails to hold in the larger class of weakly symmetric games.

Keywords: Infinite games; $\epsilon$-equilibrium; Strongly symmetric games.

JEL Codes: C70; C72.

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1 Introduction

Many interactions among a large number of participants have the following features. From the standpoint of each individual player, the population of other players is conceived as some large set, the identities of the members of which are not important. Each player’s preferences depend on (a) his own action, and (b) the actions of others. The important point regarding (b) is that the player does not care who is doing exactly what—he only cares about some crude statistics of the distribution of actions in the population.

As a simple illustration, think of tax payments: each individual player cares about (a) his own payments, and (b) the state of affairs in the economy—whether the target-revenue is being met, whether most citizens pay their taxes, and so on; he is oblivious, however, to who is paying what.

The above situations can naturally be modeled as games with countably many players. The infinity of players makes it possible to express social states (e.g., “it is customary to pay taxes”) that depend on the behavior of a large subset of players—an infinite subset, to be precise—the identity of the members of which is not important. This feature motivates the following definition: a strongly symmetric game is such that all players (1) have the same set of pure strategies, and (2) have the same utility function, $u$, where $u(a) = u(a_i, a_{-i})$ has two arguments—own-action and the actions of others—and it is invariant to all permutations on $a_{-i}$. Below I study games with countably many players, with a special emphasis on strongly symmetric ones. In all the games to be studied—strongly symmetric or not—the population of players is the only infinite object; the pure strategy sets are finite, hence the games represent a relatively moderate departure from the finite-game benchmark.

Utility-discontinuity pop up naturally in strongly symmetric games. Think of the tax payments story from above: one can get from “noone pays taxes” to “everybody pays taxes” by a sequence of steps where in each step only one person’s action is changed; therefore, if finitely many changes do not affect the state of affairs in the
grand economy (in terms of what is being taken into account by the utility function) then the utility function is either constant or discontinuous in the actions of others. For this reason, utility-discontinuity is the rule in the present paper, not the exception.

Because of this discontinuity a Nash equilibrium may fail to exist in strongly symmetric games. When Nash equilibrium fails to exist, it is natural to resort to $\epsilon$-equilibrium (due to Radner 1980): a strategy profile such that no player can increase his expected payoff by more than $\epsilon$ through a unilateral deviation. Since $\epsilon$-equilibrium does not require exact utility-maximization it is somewhat unsatisfactory, and it is therefore sensible to try to improve it by imposing on it some restrictions; in other words, to apply to it a refinement. The literature’s main refinement of $\epsilon$-equilibrium is strong $\epsilon$-equilibrium, due to Solan and Vielle (2001): a strategy profile under which each player assigns strictly positive probabilities only to $\epsilon$ best-responses (in Solan and Vielle (2001) this refinement is called perfect $\epsilon$-equilibrium; in the computer science literature it is called well-supported $\epsilon$-equilibrium; see Daskalakis et al. (2006), Goldberg and Papadimitriou (2006)). Unfortunately, strong $\epsilon$-equilibrium may fail to exist in a strongly symmetric game. Moreover, there is an important sub-class of these games—strongly symmetric tail function games—in which strong $\epsilon$-equilibria not only may fail to exist, but in which the existence of those is equivalent to the existence of Nash equilibria (and the latter may fail to exist).

Is a strongly symmetric game guaranteed to have an $\epsilon$-equilibrium, albeit not a strong one? Is there some alternative refinement of $\epsilon$-equilibrium to which strongly symmetric games adhere? I answer both these questions affirmatively.

A strategy profile is monotonic if each player assigns higher probabilities to better actions. The main positive result of this paper, Theorem 1, is that every strongly symmetric game has a monotonic $\epsilon$-equilibrium, for every $\epsilon > 0$. Theorem 1’s proof is constructive: for each $\epsilon > 0$ I construct a monotonic $\epsilon$-equilibrium. This $\epsilon$-equilibrium is symmetric, namely all players play the same strategy in it. Therefore, a corollary
obtains—in a strongly symmetric game a symmetric $\epsilon$-equilibrium exists for each $\epsilon > 0$.

The definition of strong symmetry has a straightforward weakening: a game is weakly symmetric if all players (1) have the same set of pure strategies, and (2) have the same utility function, $u$, where $u(a) = u(a_i, a_{-i})$ has two arguments—own-action and the actions of others—and it is invariant to finite permutations on $a_{-i}$. The abovementioned corollary does not apply to weakly symmetric games: I construct a weakly symmetric game in which no symmetric $\epsilon$-equilibrium exists for sufficiently small $\epsilon$'s. This non-existence is non-trivial, since the aforementioned game has infinitely many (non-symmetric) Nash equilibria.\footnote{Several results from the existing literature illustrate that symmetric games may admit only asymmetric equilibria. The first to show this was Fey (2012), who constructed symmetric 2-person games with infinite strategy sets and only asymmetric equilibria. Xefteris (2015) recently demonstrated the existence of such a zero-sum game with more than two players. In Rachmilevitch (2015) I showed a similar result for a game with finitely many strategies and a countable player set.}

The idea that players may make small mistakes, but that, despite these mistakes, better courses of action are more likely than worse ones, has been with game theory for quite a while. Building on the seminal work of Selten (1975), Myerson (1978) introduced the concept $\epsilon$ proper equilibrium. A strategy profile is such an equilibrium if (i) each player assigns a strictly positive probability to each of his pure strategies, and (ii) the following holds for each player: if the pure strategy $x$ is a better-reply than $y$, then $y$ is played with probability that is at most $\epsilon$ times the probability of $x$. The monotonicity concept considered in the present paper is a weakening of (ii) (note that the key parameter, $\epsilon$, plays a different role in the present paper and in Myerson (1978): here it refers to payoffs, whereas in Myerson’s work it refers to strategies). An alternative formalization of the idea that despite the absence of exact optimality better behavior is likelier than worse behavior can be found in McKelvey and Palfrey’s (1995) quantal response equilibrium, where the players play best-responses, but not with respect to the true utilities, but with respect to some perturbations thereof.
The rest of the paper is organized as follows. Section 2 describes preliminaries. Section 3 considers strongly symmetric games; the main result of this Section is Theorem 1; the secondary result is Corollary 1, claiming the existence of symmetric \( \epsilon \)-equilibria in strongly symmetric games. Section 4 shows that the corollary cannot be extended to weakly symmetric games. Section 5 investigates the relationship between strong and monotonic \( \epsilon \)-equilibria in general games (with countably many players and finitely many strategies). The refinements are independent: a game may have a monotonic \( \epsilon \)-equilibrium for every \( \epsilon > 0 \), but no strong \( \epsilon \)-equilibrium for small enough \( \epsilon \)'s; similarly, a game may have a strong \( \epsilon \)-equilibrium for every \( \epsilon > 0 \), but no monotonic \( \epsilon \)-equilibrium for small enough \( \epsilon \)'s. Moreover, a game may have a strong \( \epsilon \)-equilibrium for every \( \epsilon > 0 \), a monotonic \( \epsilon \)-equilibrium for every \( \epsilon > 0 \), but no \( \epsilon \)-equilibrium that satisfies both refinements when \( \epsilon \) is small enough. Finally, if a game has an \( \epsilon \)-equilibrium which is both strong and monotonic, for every arbitrarily small \( \epsilon > 0 \), this does not imply that that game has a Nash equilibrium.

2 Preliminaries

A game is a tuple \( G = [N, (A_i)_{i \in N}, (u_i)_{i \in N}] \). \( N \) is the set of players. For each \( i \in N \), \( A_i \) is the set of player \( i \)'s pure strategies (or actions), which is some finite non-empty set. The set of \( i \)'s mixed strategies is the set of probability distributions on \( A_i \), denoted \( \Delta(A_i) \). A typical element of \( \Delta(A_i) \) is denoted \( \alpha_i \). Let \( (\times_{i \in N} A_i, \mathcal{A}_P) \) be the product measurable space \( \times_{i \in N}(A_i, 2^{A_i}) \). That is, \( \mathcal{A}_P \) is the product \( \sigma \)-algebra on \( \times_{i \in N} A_i \) generated by cylinder sets of the form \( \times_{i \in N} S_i \), where \( S_i \subseteq A_i \) for all \( i \in N \) and \( S_i = A_i \) for all but finitely many \( i \)'s. Each player \( i \) has a measurable utility function \( u_i \) on \( (\times_{i \in N} A_i, \mathcal{A}_P) \), which is integrable w.r.t each product measure \( \alpha \in \times_{i \in N} \Delta(A_i) \) on \( \mathcal{A}_P \). Below I consider games \( G \) where \( N = \mathbb{N} \).

A game \( G \) is strongly symmetric if (1) all players share the same strategy set, \( A \), and (2) there exists a function, \( u \), such that (i) \( u_i(a) = u(a_i, a_{-i}) \) for all \( i \) and
a ∈ A^N, and (ii) u is invariant under all permutations on its second argument. It is a **strongly symmetric tail function game** if, additionally, the utility u satisfies the following condition: for all x ∈ A and a_{-i}, b_{-i} ∈ A^{N\setminus\{i\}}, if |{j : a_j \neq b_j}| < ∞, then u(x, a_{-i}) = u(x, b_{-i}). The tail function property expresses the idea that a “small” change in the behavior of the other players does not affect one’s utility. A game G is **weakly symmetric** if the abovementioned (1) holds, and if all players have the same utility function, as in (2), but its invariance is only guaranteed for finite permutations on its second argument.

Player i’s expected utility under α is denoted U_i(α). The action a_i is an ϵ **best-response** against α_{-i} if U_i(a_i, α_{-i}) + ϵ ≥ U_i(a'_i, α_{-i}) for all a'_i ∈ A_i \ {a_i}; it is a **best-response** if the above requirement holds for ϵ = 0. The strategy α_i is ϵ **optimal** for i against α_{-i} if U_i(α) + ϵ ≥ U_i(α'), where α' is any alternative profile that satisfies α'_j = α_j for all j ∈ N \ {i}. The profile α is an ϵ-**equilibrium** if α_i is ϵ optimal against α_{-i} for all i; it is a **Nash equilibrium** if the above requirement holds for ϵ = 0. The profile α is a **strong ϵ-equilibrium** (Solan and Vielle 2001) if for each i the following holds: α_i(a_i) > 0 implies that a_i is an ϵ best-response (for i, against α_{-i}). The strategy profile α is **pure** if for each i there is an a_i ∈ A_i such that α_i(a_i) = 1. Note that every pure ϵ-equilibrium is a strong ϵ equilibrium. The strategy profile α is **monotonic** if the following is true for every i ∈ N and x, y ∈ A_i: U_i(y, α_{-i}) > U_i(x, α_{-i}) implies α_i(y) ≥ α_i(x). The strategy profile α is a **monotonic ϵ-equilibrium** if it is an ϵ-equilibrium which is monotonic (that a profile α is monotonic does not, by itself, imply that it is an ϵ-equilibrium; for example, the profile under which each player randomizes uniformly over all his pure strategies is trivially monotonic, no matter the utility numbers it generates).

In a (weakly or strongly) symmetric game, an equilibrium (Nash or ϵ), α, is **symmetric** if α_i = α_j for all i, j ∈ N.

I make use of two central results from probability theory. The Borel-Cantelli Lemma (Billingsley 1995, p.59-60) implies that given a sequence of 0-1 independent
random variables, \( \{V_i\} \), the probability of the event \(|\{i : V_i = 1\}| < \infty \) is one if the expectation of \( \sum V_i \) is finite and is zero otherwise. A tail event is an event which depends on a countable sequence of independent random variables, which is invariant to the realization of any finite numbers of them. Kolmogorov’s 0-1 Law (Billingsley 1995, p.259) states that the probability of a tail event is either zero or one.

3 Strongly symmetric games

Peleg (1969) was the first to demonstrate that a game with infinitely many players may not have a Nash equilibrium, even if all its strategy sets are finite. The game he constructed has the pure strategy sets \( A_i = \{0, 1\} \) for all \( i \), and the utilities are:

\[
 u_i(a) = \begin{cases} 
 a_i & \text{if } \sum_{j=1}^{\infty} a_j < \infty \\
 -a_i & \text{otherwise}
\end{cases}
\]

Since, given a fixed strategy profile, \( \sum_{j=1}^{\infty} a_j < \infty \) is a tail event, its probability, call it \( p \), is either zero or one. If \( p = 1 \) then the unique best-response of each player \( i \) is to play \( a_i = 1 \), which implies \( p = 0 \). If, on the other hand, \( p = 0 \), then each \( i \)'s unique best-response is to play \( a_i = 0 \), which implies \( p = 1 \). Essentially the same argument implies that this game does not have a strong \( \epsilon \)-equilibrium for any \( \epsilon \in (0, 1) \). Since this game is a strongly symmetric game, it follows that a strongly symmetric game need not have a strong \( \epsilon \)-equilibrium.

Peleg’s game is not only strongly symmetric, but, moreover, it is a tail function game. The fact that it fails to have not only Nash equilibria, but also strong \( \epsilon \)-equilibria, is a consequence of a general result.

**Proposition 1.** Let \( G \) be a strongly symmetric tail function game. Then the following are equivalent:

1. \( G \) has a strong \( \epsilon \)-equilibrium for all \( \epsilon > 0 \).

2. \( G \) has a Nash equilibrium.
Proof. I will prove that 1 implies 2. Let $G$ be such a game. Let $A$ and $u$ be its strategy set and utility function. For each $\epsilon > 0$ fix a strong $\epsilon$-equilibrium, $\alpha^{\epsilon}$. By the 0-1 Law, precisely one of the following is true for each $x \in A$: (a) the probability of $|\{j : a_j = x\}| = \infty$ under $\alpha^{\epsilon}$ is one, or (b) the probability of $|\{j : a_j = x\}| = \infty$ under $\alpha^{\epsilon}$ is zero. Clearly, there are $x$’s for which (a) applies. Denote the set of those by $A(\epsilon)$. Since $A$ is finite, there is a $B \subset A$ such that for every $\epsilon > 0$ there is an $\epsilon' \in (0, \epsilon)$ such that $A(\epsilon') = B$.

Let $E$ be the event “each action in $B$ is played infinitely many times.” I argue that each $b \in B$ is a best-response for $i$ against $E$. To see this, assume by contradiction that there is a $b^* \in B$ which is not. Then there is an $x \neq b^*$ such that $\delta > 0$, where $\delta \equiv u(x,e) - u(b^*,e)$, where $e$ is any element of $E$. Consider an $\epsilon \in (0, \delta)$ such that $A(\epsilon) = B$. The action $b^*$ is played infinitely many times with probability one under $\alpha^{\epsilon}$ and it is therefore $\epsilon$-optimal against $E$, therefore $\delta < \epsilon$—a contradiction.

Since each $b \in B$ is a best-response for $i$ against $E$, it follows that a uniform randomization over $B$ is a Nash equilibrium strategy.

The tail function property is important—a strongly symmetric game that has a strong $\epsilon$-equilibrium for all $\epsilon > 0$ but whose utility function is not a tail function may fail to have a Nash equilibrium (see Proposition 1 in Rachmilevitch (2015)).

The following result shows that, in contrast to strong $\epsilon$-equilibria, monotonic $\epsilon$-equilibria are guaranteed to exist in a strongly symmetric game (no matter whether it is a tail function game or not).

Theorem 1. Let $G$ be a strongly symmetric game and let $\epsilon > 0$. Then $G$ has a monotonic $\epsilon$-equilibrium.

Proof. Let $G$ be a game as above; denote its common utility function and strategies set by $u$ and $A$. Let $\epsilon > 0$. If $G$ has a Nash equilibrium then there is nothing to prove. Suppose, then, that it does not have a Nash equilibrium. Let $E$ be the event “each $a \in A$ occur infinitely many times.” Let $A^*$ be the set of best-responses
against $E$. By assumption, $A^* \subsetneq A$ (or else $G$ would have a Nash equilibrium). Let 
\[ \delta \equiv u(a^*, e) - \min_{a \in A} u(a, e), \] 
where $e$ is any element of $E$. By strong symmetry, $\delta$ is well defined. Let $\epsilon' \in (0, \min\{\epsilon, \frac{\delta(A \setminus A^*)}{|A|}\})$. The profile where each player assigns probability $\frac{\epsilon'}{\delta(A \setminus A^*)}$ to each element of $A \setminus A^*$ and spreads the remaining probability mass uniformly over $A^*$ is a monotonic $\epsilon'$-equilibrium. Therefore, it is also a monotonic $\epsilon$-equilibrium. 

The following is an immediate consequence of Theorem 1’s proof.

**Corollary 1.** Let $G$ be a strongly symmetric game and let $\epsilon > 0$. Then $G$ has a symmetric $\epsilon$-equilibrium.

## 4 Weakly symmetric games

In Corollary 1 “strongly symmetric” cannot be weakened to “weakly symmetric.” Moreover, existence of a symmetric $\epsilon$-equilibrium is not guaranteed even if the weakly symmetric game in question has Nash equilibria.

**Proposition 2.** There exists a weakly symmetric game that has a Nash equilibrium but does not have a symmetric $\epsilon$-equilibrium, for any $\epsilon \in (0, \frac{1}{2})$.

To describe the game whose existence is asserted in Proposition 2, some preliminary definitions are needed. Given a number $r$ and a positive integer $m$, let $r_m$ be the $m$-dimensional vector with all coordinates equal to $r$. Say that a profile $a \in \{0, 1\}^N$ has the increasing tail property (ITP for short) if there exist $K, m \in \mathbb{N}$ such that:

\[ a = (a_1, \cdots, a_K, 1_m, 0_{m+1}, 0_{m+1}, 1_{m+2}, 0_{m+2} \cdots). \]

Let $v^*_i: \{0, 1\}^N \rightarrow \{0, 1\}$ be the following function:

\[ v^*_i(a) = \begin{cases} 
1 & \text{if } (\limsup_{k \to \infty} \frac{a_1 + \cdots + a_k}{k} > \frac{1}{2} \text{ and } a_i = 0) \text{ or } (\limsup_{k \to \infty} \frac{a_1 + \cdots + a_k}{k} \leq \frac{1}{2} \text{ and } a_i = 1) \\
0 & \text{otherwise}
\end{cases} \]

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Voorneveld (2010, Example 4.3) proved that the game whose strategy set and utility functions are \(\{0, 1\}\) and \((v^*_i)_{i \in \mathbb{N}}\) does not have an \(\epsilon\)-equilibrium, for any \(\epsilon \in (0, \frac{1}{2})\).

**Proof of Proposition 2**: Let \(G\) be the weakly symmetric game in which the strategy set is \(\{0, 1\}\) and the utilities are:

\[
u_i(a) = \begin{cases} 1 & \text{if } a \text{ has ITP} \\ v^*_i(a) & \text{otherwise} \end{cases}
\]

It is easy to see that \(G\) has Nash equilibria: every ITP profile is such an equilibrium. It remains to prove that \(G\) does not have a symmetric \(\epsilon\)-equilibrium, for \(\epsilon \in (0, \frac{1}{2})\). Assume by contradiction that such an equilibrium exists, \(\alpha\). In view of Voorneveld’s result, it is enough to show that the probability of ITP (under \(\alpha\)) is zero. This is clear if \(\alpha\) is a profile under which all players take the same pure action; so suppose that each player takes the action 1 with probability \(q\) for some \(q \in (0, 1)\). Since the event \(E = \{a \text{ has ITP}\}\) is a tail event, its probability is either zero or one. It cannot be one, because the probability of \(\{0, 1\}^\mathbb{N} \setminus E\) is at least as large as the probability of \(E\). \(\square\)

## 5 Monotonic vs. strong \(\epsilon\)-equilibria

As we saw in Peleg’s game, the existence of monotonic \(\epsilon\)-equilibrium does not imply the existence of strong \(\epsilon\)-equilibrium. In the other direction, a game may have strong \(\epsilon\)-equilibria but no monotonic ones.

**Proposition 3.** There exists a game that has a strong \(\epsilon\)-equilibrium for every \(\epsilon > 0\), but does not have a monotonic \(\epsilon\)-equilibrium for any \(\epsilon > 0\).

**Proof.** Consider the following game, in which \(A_i = \{0, 1\}\) for all \(i\), and the utility functions are:
Let 

\[ u_i(a) = \begin{cases} 
-2^{-\sum_{j=1}^{\infty} a_j} & \text{if } \sum_{j=1}^{\infty} a_j < \infty \\
-i^2 \cdot a_i & \text{otherwise} 
\end{cases} \]

**Strong \( \epsilon \)-equilibrium.** Fix an \( \epsilon > 0 \). Consider the following profile: all \( i \leq K \) play the pure action 1 and any other player plays the pure action 0, where \( K \) satisfies \( 2^{-K-1} < \epsilon \). It is easy to check that this is an \( \epsilon \)-equilibrium; since it is pure, it is strong.

**Non-existence of monotonic \( \epsilon \)-equilibrium.** Fix an \( \epsilon > 0 \). Assume by contradiction that \( \alpha \) is such an equilibrium. By Kolmogorov’s 0-1 Law the probability of \( \sum_{j=1}^{\infty} a_j < \infty \), call it \( p \), satisfies \( p \in \{0, 1\} \).

If \( p = 1 \) then the unique best-response of each player is the action 1, and hence, by monotonicity, each player plays 1 with probability which is bounded from below by one half; therefore, by the Borel-Cantelli Lemma, \( p = 0 \).

If \( p = 0 \) then, by \( \epsilon \) optimality, each player \( i \) plays the pure action 1 with probability which is no greater than \( \frac{\epsilon}{i^2} \), and this, by the Borel-Cantelli Lemma, implies \( p = 1 \). \( \square \)

The two refinements, “monotonic” and “strong,” can be combined into a stronger refinement: \( \alpha \) is a **monotonic strong \( \epsilon \)-equilibrium** if it is a strong \( \epsilon \)-equilibrium and is monotonic. The following result illustrates the strength of this joint refinement.

**Proposition 4.** There exists a game \( G \) with the following properties:

- \( G \) has a monotonic \( \epsilon \)-equilibrium for every \( \epsilon > 0 \).
- \( G \) has a strong \( \epsilon \)-equilibrium for every \( \epsilon > 0 \).
- There does not exist an \( \epsilon \in (0, 1) \) such that \( G \) has a monotonic strong \( \epsilon \)-equilibrium.

**Proof.** Consider the following game, in which the strategy set of player \( i \) is \( A_i = \{1, \cdots, i\} \), and the utilities are:
Monotonic $\epsilon$-equilibrium. Fix an $\epsilon > 0$. Wlog, suppose that $\epsilon < \frac{1}{2}$. Let player 1 play the pure action 1. Let each player $i \geq 2$ play the action 1 with probability $1 - \epsilon$ and play each of his other actions with probability $\frac{\epsilon}{i-1}$. By the Borel-Cantelli Lemma, the probability of $|\{j : a_j = j\}| < \infty$ under this profile is zero; hence, this is an $\epsilon$-equilibrium. It is a monotonic such equilibrium because better (i.e., lower-indexed) actions are being assigned (weakly) higher probabilities.

Strong $\epsilon$-equilibrium. Fix an $\epsilon > 0$ and consider the following profile: fix a $K$ such that $\frac{1}{K} < \epsilon$, let each player $i \geq K$ play his “second-best” pure strategy, namely $(i - 1)$, and let all $i < K$ play their best-response, namely $a_i = i$. It is easy to see that this is a strong $\epsilon$-equilibrium.

Non-existence of monotonic strong $\epsilon$-equilibrium. Fix an $\epsilon \in (0, 1)$ and assume by contradiction that $\alpha$ is a monotonic strong $\epsilon$-equilibrium. By Kolmogorov’s 0-1 Law, the probability of $|\{j : a_j = j\}| < \infty$ under $\alpha$, call it $p$, satisfies $p \in \{0, 1\}$. If $p = 0$ then, since $\alpha$ is a strong $\epsilon$-equilibrium, each $i > \frac{1}{1 - \epsilon}$ assigns probability zero to $a_i = i$; this implies that $p = 1$—a contradiction. If, on the other hand, $p = 1$, then by the monotonicity requirement, $\alpha_i(i) \geq \frac{1}{i}$; by the Borel-Cantelli Lemma, $p = 0$. □

Despite its strength, monotonic strong $\epsilon$-equilibrium does not guarantee the existence of Nash equilibrium.

Proposition 5. There exists a game that has a monotonic strong $\epsilon$-equilibrium for every $\epsilon > 0$, but does not have a Nash equilibrium.

Proof. Let $2\mathbb{N}$ denote the even natural numbers. Let:

- $A_{\text{odd}} \equiv \{ a \in \{0, 1\}^{\mathbb{N}} : \sum_{j \in 2\mathbb{N}} a_j < \infty = \sum_{j \not\in 2\mathbb{N}} a_j \}$,

- $A_{\text{even}} \equiv \{ a \in \{0, 1\}^{\mathbb{N}} : \sum_{j \not\in 2\mathbb{N}} a_j < \infty = \sum_{j \in 2\mathbb{N}} a_j \}$,
\[ A_\infty \equiv \{ a \in \{0, 1\}^N : \sum_{j \in 2N} a_j = \sum_{j \notin 2N} a_j = \infty \}, \]

and

\[ A_{\text{finite}} \equiv \{ a \in \{0, 1\}^N : \sum_{j \in 2N} a_j, \sum_{j \notin 2N} a_j < \infty \}. \]

Consider the following game, in which \( A_i = \{0, 1\} \) for all \( i \in \mathbb{N} \). On \( A_{\text{odd}} \cup A_{\text{even}} \), utilities are defined as follows:

\[
u_i(a) = \begin{cases} 
a_i & \text{if } a \in A_{\text{odd}} \text{ and } i \text{ is even} \\
-a_i & \text{if } a \in A_{\text{even}} \text{ and } i \text{ is even} \\
a_i & \text{if } a \in A_{\text{even}} \text{ and } i \text{ is odd} \\
-a_i & \text{if } a \in A_{\text{odd}} \text{ and } i \text{ is odd} 
\end{cases}
\]

Given \( A_{\text{finite}} \) each player \( i \)'s utility equals his action, \( a_i \). Finally, given \( A_\infty \), the player population is partitioned into the pairs \( \{ (i-1, i) : i \in 2\mathbb{N} \} \) and payoffs for each pair are determined by a 2-person game for this pair. Given \( a \in A_\infty \), each \( (i-1, i) \) receive payoffs as follows:

\[
\begin{array}{c|cc|cc}
i - 1, i & 0 & 1 \\
\hline
0 & \frac{1}{i_1}, \frac{1}{i_1} & \frac{1}{i_1}, 0 \\
1 & 0, \frac{1}{i_1} & 0, 0
\end{array}
\]

**Monotonic strong \( \epsilon \)-equilibrium.** Fix an arbitrary \( \epsilon > 0 \). Let \( N^* \) be the smallest odd integer which is weakly greater than \( \frac{1}{\epsilon} \). Consider the following profile: each \( i < N^* \) plays the action 0, and each \( i \geq N^* \) plays 0 with probability \( \lambda \), where \( \lambda \in (\frac{1}{2}, 1) \). Note that this is a monotonic strong \( \epsilon \)-equilibrium.

**Non-existence of Nash equilibrium.** Assume by contradiction that \( \alpha \) is a Nash equilibrium. By the 0-1 Law, the probability of \( A_{\text{odd}} \) under \( \alpha \) is either zero or one. It cannot be one, since in this case each even \( i \) has a unique best-response—to play the action 1. By the analogous arguments, the probability of \( A_{\text{even}} \) is also zero. Since
$\mathcal{A}_\infty$ and $\mathcal{A}_{\text{finite}}$ are tail events, exactly one of them occurs under $\alpha$ with probability one. Let $p$ denote the probability of $\mathcal{A}_\infty$. If $p = 1$ then the unique best-response of each player is to play the action 0, which implies $p = 0$. If $p = 0$, then the unique best-response of each $i$ is to play 1, which implies $p = 1$.

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