Stability in Large Bayesian Games with Heterogeneous Players

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Abstract

Bayesian Nash equilibria that fail to be hindsight-(or, alternatively, ex-post) stable do not provide reliable predictions of outcomes of games in many applications. We characterize a family of large Bayesian games (with many players) in which all equilibria are asymptotically hindsight-stable, and discuss the consequences of this robustness property. In contrast to earlier literature, we establish hindsight stability in a class of games in which players are not anonymous and type spaces and action spaces can be infinite.

*Acknowledgements. The authors would like to thank Michael Kearns, Colin McDiarmid, Eran Shmaya, an associate editor and two anonymous referees for useful comments and suggestions. We also thank seminar participants at GAMES 2008, Institut Henri Poincaré, Northwestern University, SAET meetings in Singapore and Tel Aviv University for helpful comments. This is a substantially revised version of Chapter 3 of the first author’s Ph.D. dissertation submitted to Northwestern University in June 2008.

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1 Introduction

Hindsight stability, also referred to as ex-post robustness, is an important property of equilibria of one-shot simultaneous-move Bayesian games. An equilibrium of a Bayesian game is hindsight-stable if no player has an incentive to change her action after she learns the realized types and the realized actions of all the players. Equilibria that are not hindsight-stable may not provide useful predictions of outcomes in games in which players can revise choices based on hindsight information; and a significant number of social interactions are, indeed, of this type (see Section 1.1 for examples).

There is a substantial literature devoted to issues of stability in games with many players, or large games. Kalai (2004) shows that all equilibria in a certain class of large games are structurally robust, a stronger property that implies hindsight stability. However, the restrictions on the class of games in Kalai (2004) leave out many important real-life situations. The first restriction is that the number of possible types and actions of the players be finite. Second, when a player evaluates her payoffs, she is restricted to viewing all her opponents as anonymous or indistinguishable. Both these assumptions severely limit the applicability of the results.

The main objective of this paper is to characterize an important family of large games not restricted by assumptions of finiteness and anonymity and, yet, in which all equilibria are still hindsight-stable.

We consider a general family of Bayesian games in which players’ types and action spaces are compact subsets of finite dimensional Euclidean spaces, and in which players’ types are drawn independently. We assume that payoffs can depend on types and actions of specific rivals in asymmetric ways, but subject to two regularity conditions: “Uniform Lipschitz Continuity in one’s own types and actions”, and “Uniform Scaled Lipschitz Continuity in rivals’ types and actions.” In the main result of this paper, we show that these two conditions are sufficient to guarantee that all equilibria of these games become approximately hindsight-stable at an exponential rate as the number of players increases.

The regularity conditions we impose on the payoff functions are interesting in their own right. In particular, the Uniform Scaled Lipschitz continuity with respect to rivals implies that a player has a limited individual impact on her opponents and that the impact decreases with the total number of players in the game. Different bounds on the informational size or influence of players have been presented before. See, for example, Al-Najjar and Smorodinsky (2000), McLean and Postlewaite (2002) and references therein.
impact does not imply anonymity and, as we illustrate in examples below, is far from assuming that players become approximately symmetric.

From a mathematical point of view, the limited individual impact condition allows us to apply a powerful law of large numbers: McDiarmid’s Bounded Differences Inequality. This law and its variants may be useful in a variety of economic settings.\(^3\)

The proof of the main result is done in three steps. In the first step, we establish a basic relationship between realized payoffs and expected payoffs in a Bayesian game: In particular, we use McDiarmid’s Inequality to prove that the realized payoff of any player from playing a particular strategy is close to the ex-ante expected payoff from that strategy, with a high probability, as long as the number of players is large. Second, we use this result to establish a result about convergence to hindsight stability in games with a finite number of possible types and actions. Finally, by approximating a general (infinite) game by a finite counterpart (its finite grid reduction game), we show that the convergence to hindsight stability holds in general.

Hindsight stability relates to other important issues in economic applications: It implies that mixed strategies self-purify (see Kalai (2004) and Cartwright and Wooders (2009)); it implies a strong rational expectations property in certain market games (see Kalai (2004, 2008)); and it implies that the revelation principle holds in implementation problems (see Green and Laffont (1987)). Robustness properties of equilibria are of interest in computer science, in the areas of distributed communications and computing (see Halpern (2003)). In particular, robust equilibria perform better in systems that involve asynchronous communications (see Kalai (2005)), and in protocols that involve faulty behavior (see Gradwohl and Reingold (2008)).

Related current work on equilibrium robustness in large games includes Gradwohl and Reingold (2010), who study robustness of Bayesian equilibria in games that allow certain correlations between players’ types.\(^4\) Carmona and Podczeck (2010) present a result on approximate ex-post stability of Bayesian equilibria in large games with infinite type and action spaces but, unlike us, they assume anonymity. However, they allow for infinite dimensional type and action spaces, and for discontinuity in one’s own type and action. Other recent work on large games includes Azrieli and Shmaya (2010), who study purification in large non-anonymous complete-information games, and Bodoh-Creed (2010), who studies implementation in large games.

\(^3\)Unlike classical laws of large numbers, which deal with the expected value of the average of random variables, this law deals with the expected value of any function of the variables in which the impact of individual variables is limited.

\(^4\)They obtain a result with limited hindsight stability for non-anonymous players.
1.1 Illustrative Examples

Before we go into the formal model and results, we first present some examples that illustrate the results we have about hindsight stability in games.

**Example 1. A Location game with men and women:** Simultaneously, \( n \) male players, \( m_1, m_2, ..., m_n \), and \( n \) female players, \( f_1, f_2, ..., f_n \), choose locations, \( l_{m_j} \) and \( l_{f_i} \), in the interval \([0, 1]\). Men want to be close to the women, but women want to be far from the men. Specifically, each female player’s utility is \( u_{f_i} = \frac{1}{n} \sum_{j=1}^{n} |l_{f_i} - l_{m_j}| \), whereas each male player’s utility is \( u_{m_i} = 1 - \frac{1}{n} \sum_{j=1}^{n} |l_{m_i} - l_{f_j}| \).

At a symmetric equilibrium of this game, all the men choose to locate at \( \frac{1}{2} \), and each woman chooses between 0 and 1 with equal probability.\(^6\) Clearly, with a small, odd number of players, such an equilibrium is not hindsight-stable. Our main result implies that asymptotically, as \( n \) becomes large, all equilibria of the location game become hindsight-stable. Specifically, for an arbitrarily small positive number \( \varepsilon \), the probability of the event that “some player can gain more than \( \varepsilon \) by unilaterally revising her action ex-post” decreases to zero at an exponential rate as \( n \) becomes large.

In the next example, we consider a game with heterogeneous players.

**Example 2. Heterogeneous payoff functions.** Consider a location game as above, but with players \( g_i, g = 1, 2, ..., G \), for some fixed integer \( G \), and \( i = 1, 2, ..., n \) (\( nG \) players in total). The index \( g \) denotes groups, such as genders, tribes, nationalities, races or combinations of such characteristics, and \( i \) names the player within a group. Each player chooses a point \( l_{g_i} \) in a set \( C_{g_i} \), subset of some fixed compact set in \( \mathbb{R}^m \).

Players have heterogeneous payoff functions that depend on the identity of their specific opponents. A simple example may be a player \( 5_3 \) who values the proximity to other players in a exponentially decreasing order in group similarity: \( u_{5_3} = 3 - \sum_{g=1}^{G} \left( \frac{1}{2} \right)^{|g-3|} \sum_{i=1}^{n} \frac{1}{n} d(l_{5_3}, l_{g_i}) \), where \( d(x, y) \) is the distance between \( x \) and \( y \).

The payoff function \( u_{5_3} \) satisfies the regularity conditions required in our setting. In particular, the rate of change of \( u_{5_3} \), as one changes the location of any one opponent, decreases to zero as the number of players increases. Thus, players have limited individual impact. However, while individual opponents become of negligible importance, they are far from being equally unimportant. No matter how large \( n \) is, for player \( 5_3 \), the average location of the players in group 5 is twice as important as that

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\(^5\)This is a generalization of the Village vs. Beach game (Kalai (2004)) with a continuum of actions.

\(^6\)At other equilibria, all the men choose \( \frac{1}{2} \), and every woman chooses, purely or randomly, one of the two extremes. For odd \( n \), some females must randomize.
of players in group 6, which is twice as important as that of players in group 7, etc. Our main result implies that in this example, as the number of players increases, all equilibria become hindsight-stable.

It is worthwhile to note that, as stated, Example 2 can still be re-cast as a semi-anonymous game and viewed as a special case of Kalai (2004). However, this is no longer true if the number of groups becomes infinite (or increases with \( n \)).

**Example 3.** Consider the same game as in Example 2 with players \( g_i \), but with the number of possible groups \( g = 1, 2, ..., n \), so that there are \( n^2 \) players. Suppose that the payoff function for player \( 53 \) is:

\[
 u_{53} = \sum_{g=1}^{n}(\frac{1}{2})^{g-5}\sum_{j=1}^{n}\frac{1}{n^2}d(l_{53} - l_{g}).
\]

Such a game satisfies the conditions in this paper, but cannot be accomodated in the earlier Kalai model. The heterogeneity of types allowed in our setting goes significantly further than in the example above. For instance, the payoff functions need not treat opponents within a group symmetrically.

**Example 4. Bayesian Cournot game:** The players are \( n \) sellers, each having the capacity to produce up to \( k \) units of an identical divisible product. The price-quantity relationship is described by the demand function: \( p = 1 - q/n \). Each seller \( i \) knows his per-unit production costs \( c_i \)--i.e., his type, which is randomly drawn by a commonly known prior probability distribution \( \mu_i \) over some fixed interval of real numbers \( I_i \).

With knowledge of his own type, each player decides on a production level \( 0 \leq x_i \leq k \), resulting in a profile \((c, x)\), of individual costs and quantities. The resulting payoff of player \( i \) is

\[
 u_i(c, x) = x_i(1 - \sum_{j=1}^{n} x_j/n - c_i).
\]

In Bayesian Cournot games, pure strategy equilibria may not exist. Even when an equilibrium exists, if the number of players is small, the quantity choices are likely to be hindsight-unstable. Our result establishes that if there are a large number of producers (and buyers), any Bayesian equilibrium is approximately hindsight-stable.

Two additional properties are direct consequences of our result. First, hindsight-stability means that the realized pure actions will constitute an (approximate) equilibrium of the complete-information game determined by the realized types. In this sense, the Bayesian equilibrium is (asymptotically) self-purifying. Second, in every Bayesian equilibrium, the realized actions will satisfy a rational expectations property.

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7 A simple interpretation is that there are \( n \) buyers, each with a demand function \( q = 1 - p \).

8 See discussion and references in Einy et al. (2010).

9 This property is significantly stronger than (simple) purification, which originated the study of large strategic games. See Schmeidler (1973) and the follow-up literature. Under simple purification, one simply establishes the existence of a pure-strategy equilibrium.
to yield realized prices that are competitive. While obtaining competitive equilibrium as the limit of Cournot games is not new,\textsuperscript{10} here, it is an immediate consequence of our result in the more general context of Bayesian games. Moreover, it is not obtained at a particular equilibrium, but is the (asymptotic) property of every Bayesian equilibrium.

2 Model

Let \( \mathcal{T} \) denote the space of feasible types of players, and let \( \mathcal{A} \) denote the space of all feasible actions of players. We assume that \( \mathcal{T} \) and \( \mathcal{A} \) are fixed subsets of some Euclidean spaces. We will consider a family \( \Gamma = \Gamma(\mathcal{T}, \mathcal{A}) \) of Bayesian games \( G(N, T, A, \tau, \{u_i\}) \) that can each be described as follows.

- There are \( N \) players \( \{i = 1, ..., N\} \).
- The type of each player \( i \) is drawn independently from a type-space \( T_i \), where \( T_i \) is a compact subset of \( \mathcal{T} \). Players are informed about their own type. Let \( T \) denote the type space of all players–i.e., \( T := \prod_{i=1, ..., N} T_i \). Let \( \tau \) be a probability measure on the Borel subsets of \( T \). Under the independence assumption (assumed throughout the paper), \( \tau \) is the product measure of its marginal distributions on each \( T_i \), denoted by \( \tau_i \). The type distributions are common knowledge.
- Each player \( i \) chooses actions from her action space \( A_i \) which is a compact subset of \( \mathcal{A} \). Let \( A \) denote the space of action profiles of all players–i.e., \( A := \prod_{i=1, ..., N} A_i \).
- Players’ payoffs are given by bounded measurable functions \( u_i : \mathcal{C} \rightarrow \mathbb{R} \). For convenience, we sometimes use the derived functions \( u_i^{c_i}(c_{-i}) : \mathcal{C}_{-i} \rightarrow \mathbb{R} \) where \( u_i^{c_i}(c_{-i}) = u_i(c_i, c_{-i}) \).

**Definition 1 (Uniformly Bounded Strategy Space).** A family of games \( \Gamma = \Gamma(\mathcal{T}, \mathcal{A}) \) is said to have a uniformly bounded strategy space if \( \mathcal{T} \) is a compact subset of \( \mathbb{R}^{k_T} \), and \( \mathcal{A} \) is a compact subset of \( \mathbb{R}^{k_A} \) for some fixed integers \( k_T, k_A > 0 \).

\textsuperscript{10}See Mas-Colell (1983), Novshek and Sonnenschein (1983) and the follow-up literature.
We define the standard $L_1$-metric to be $d(x, y) = \sum_{m=1}^{M} |x_m - y_m|$ for any $x, y \in \mathbb{R}^M$. We consider families of games with uniformly bounded strategy spaces, and with payoff functions that satisfy the following two regularity conditions.

**Definition 2 (LC1: Uniform $K$-Lipschitz Continuity in One’s Own Character).**
Given $K \geq 0$, the payoff functions $u_i$ in a family of games $\Gamma(\mathcal{F}, \mathcal{A})$ are said to be uniformly $K$-Lipschitz continuous in one’s own character, if for every player $i$, any character profile $c$ and any type-action character of player $i$, $c'_i$,

$$|u_i(c_i, c_{-i}) - u_i(c'_i, c_{-i})| < K d(c_i, c'_i),$$

where $d(., .)$ is the $L_1$ metric.

Condition LC1 is a form of Lipschitz Continuity in one’s own types and actions.

**Definition 3 (LC2: Uniform Scaled $L$-Lipschitz Continuity in Rival Character Profile).**
Given $L \geq 0$, the payoff functions $u_i$ in a family of games $\Gamma(\mathcal{F}, \mathcal{A})$ are said to be Uniformly Scaled $L$-Lipschitz Continuous in the rivals’ type-action character, if for every $N$-player game in $\Gamma(\mathcal{F}, \mathcal{A})$, for every player $i$, for any $c, c'_{-i}$,

$$|u_i(c_i, c_{-i}) - u_i(c'_i, c_{-i})| < \frac{L}{N-1} d(c_{-i}, c'_{-i})$$

where $d(., .)$ is the $L_1$ metric.

Notice that the Lipschitz bound $L$ is uniform for all $N$ in the family of games.

Next, we define a strategy in this environment. Defining mixed strategies as maps from types to mixtures over pure strategies has the drawback that they are not well-defined in games with a continuum of types (see Aumann (1964)). We use the notion of distributional strategies as introduced by Milgrom and Weber (1985). A distributional strategy is simply another way of representing mixed and/or behavioral strategies.

**Definition 4 (Distributional Strategy).** A distributional strategy for player $i$ is a probability measure $\sigma_i$ on the Borel subsets of $T_i \times A_i$ for which the marginal distribution of $T_i$ coincides with $\tau_i$. Formally, for any $S \subset T_i$, $\sigma_i(S \times A_i) = \tau_i(S)$.

When players use distributional strategies, the expected payoff of player $i$ is defined as follows:

$$U_i(\sigma) = \int u_i(c) d\sigma(c),$$

where, for a profile of distribution strategies, we let $\sigma = (\sigma_1, \ldots, \sigma_N)$ denote the product distribution over $\prod(T_i \times A_i) = C$. 

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Definition 5 (Equilibrium). A profile of (distributional) strategies $\sigma^*$ is an equilibrium\textsuperscript{11} if

$$ U_i(\sigma_1^*, \ldots, \sigma_i^*, \ldots, \sigma_N^*) \geq U_i(\sigma_1^*, \ldots, \sigma_i', \ldots, \sigma_N^*). $$

Since we prove robustness properties of equilibria asymptotically, as the number of players increases, we need to define a notion of approximate equilibrium.

Definition 6 ($\varepsilon$-Best Response). Let $\varepsilon > 0$ (small). A strategy $\sigma_i^*$ is an $\varepsilon$-best response for player $i$ to $\sigma_{-i}$ if for every positive-probability set of player $i$'s types, $\tilde{T}_i \subset T_i$ (with $\tau_i(\tilde{T}_i) > 0$), the following condition holds: $U_i((\sigma_i', \sigma_{-i})|\tilde{T}_i) - U_i((\sigma_i^*, \sigma_{-i})|\tilde{T}_i) \leq \varepsilon$ for all distributional strategies $\sigma_i'$ of player $i$.

Note that here, we define the best response for a player in an interim sense, i.e. after a player observes his own realized type (as is standard in the study of equilibria of Bayesian games).

Definition 7 ($\varepsilon$-Equilibrium). A strategy profile $\sigma^*$ is an $\varepsilon$-equilibrium if each $\sigma_i^*$ is an $\varepsilon$-best response to $\sigma_{-i}$.

Finally, we introduce the appropriate notion of robustness, which we call hindsight stability. A type-action profile is hindsight-stable for a player if she does not want to change her action even after she observes the realized types and actions of her rivals. Formally, we define approximate hindsight stability as follows.

Definition 8 (Approximate Hindsight Stability). A type-action character profile $c = (t, a)$ is $\varepsilon$-hindsight-stable for player $i$ if $u_i((t_i, a_i), c_{-i}) - u_i(c) \leq \varepsilon$ for all $a_i \in A_i$.

The type-action character profile is $\varepsilon$-hindsight-stable, if it is $\varepsilon$-hindsight-stable for all players. A strategy profile $\sigma^*$ is $(\varepsilon, \rho)$-hindsight-stable if it yields $\varepsilon$-hindsight-stable type-action profiles with probability at least $1 - \rho$.

Note that an equilibrium is approximately hindsight-stable if the realized actions, not the mixed actions, constitute an approximate Nash equilibrium of the realized complete-information game.

2.1 Games with Limited Individual Impact

It is worth emphasizing that LC2 and uniform boundedness of the strategy space, together, imply a property that we call “limited individual impact (LII).” LII means that

\textsuperscript{11}In this paper, we are not concerned about existence of equilibria, since our objective is to establish stability properties of equilibria where they exist. Note that independence of types and uniform continuity of payoff functions deliver existence of an equilibrium in distributional strategies.
the effect that any player can unilaterally have on an opponent’s payoff is uniformly bounded and decreases with the number of players in the game.

**Definition 9 (λ-Limited Individual Impact).** Given $\lambda \geq 0$, players in games in $\Gamma(\mathcal{T}, \mathcal{A})$ are said to have $\lambda$-limited individual impact if the set of payoff functions $\{u_i\}$ satisfies the following condition:

For all $i$, for all type-action characters $c, c'$, $|u_i(c_i, c_{-i}) - u_i(c_i, c'_{-i})| \leq \frac{\lambda}{N - 1}$,

whenever $c_{-i}, c'_{-i}$ differ only in one coordinate.

The LII property does not imply anonymity or even asymptotic anonymity.\(^{12}\)

**Lemma 1.** Let $\Gamma(\mathcal{T}, \mathcal{A})$ be a family of games with a uniformly bounded strategy space and satisfying the Uniform Scaled $L$-Lipschitz Continuity in rival character profiles. Then, there exists a constant $\lambda \geq 0$ such that players in $\Gamma(\mathcal{T}, \mathcal{A})$ satisfy the $\lambda$-limited individual impact condition.

**Proof.** Consider any game in $\Gamma(\mathcal{T}, \mathcal{A})$, and fix any player $i$. For any two type-action character profiles of player $i$’s rivals $c_{-i}$ and $c'_{-i}$ that differ only in one coordinate, the distance between the two profiles is less than some upper bound $B$. The existence of such an upper bound follows from the uniform bounded strategy space. Now the Uniform Scaled $L$-Lipschitz Continuity (LC2) implies that for any type-action character of player $i$, the difference in player $i$’s payoffs at $c_{-i}$ and $c'_{-i}$ can be, at most, $\frac{LB}{N - 1}$. This implies that the $\lambda$-limited individual impact condition holds with $\lambda = LB$. \(\square\)

LII is technically quite useful, as it implies that a certain law of large numbers (McDiarmid’s Bounded Differences Inequality) holds. We use this to prove asymptotic convergence to hindsight stability. To the best of our knowledge, this law of large numbers has not been used before in the economics literature, and can be very useful.\(^{13}\)

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12Recent work by Azrieli (2009) on conjectural categorial equilibrium also uses the notion of individual impact being inversely proportional to the number of players. Azrieli and Shmaya (2010) define a related property called “maximal impact” in their work related to purification of equilibria in games with complete information. They establish the existence of pure approximate equilibrium in games with sufficiently small maximal impact.

13Recently, some related concentration results, like Hoeffding’s inequality, have been used in the learning literature: for instance, Cesa-Bianchi and Lugosi (2006). McDiarmid’s inequality is a further generalization.
3 Hindsight Stability in Large Games

The main result of the paper is a robustness result. If we consider a family of games satisfying our regularity conditions, then any Bayesian equilibrium in this family is approximately hindsight-stable if a large number of players play the game. The formal statement of the result is as follows:

Theorem 1 (Hindsight Stability in Large Games). Consider a family of games \( \Gamma(\mathcal{T}, \mathcal{A}) \) with a uniformly bounded strategy space and satisfying regularity conditions LC1 and LC2. Given \( \varepsilon > 0 \), there exist constants \( \alpha = \alpha(\Gamma, \varepsilon), \beta = \beta(\Gamma, \varepsilon) < 1 \) such that if \( \sigma^* \) is an equilibrium of a game \( G(N, T, A, \tau, \{u_i\}) \in \Gamma \) then \( \sigma^* \) is \((\varepsilon, \alpha\beta^N)\) hindsight-stable.

The rest of this section is devoted to proving this theorem. We do this in several steps. First, we restrict attention to a family games with a finite number of actions and types. We show that in any family of finite games satisfying condition LC2, the realized payoff of a player from any strategy must be “close” to the ex-ante expected payoff from that strategy, given the expected equilibrium play of his opponents. Using this result, we establish asymptotic hindsight stability of a particular class of approximate equilibria (that we call “strong approximate equilibria”) in this family of finite games. Finally, we use the regularity conditions LC1 and LC2 to show that a game with an infinite number of actions and types may be approximated by a finite grid reduction game, in which the number of types and actions is finite, such that equilibria of the infinite game are strong approximate equilibria in the finite grid reduction game. We then invoke our results about convergence to hindsight stability of strong approximate equilibria in finite games, to establish that equilibria of the original infinite game are also approximately hindsight-stable.

3.1 Games with Finite Type-Action Spaces

In this subsection, we restrict attention to games with finite type and action spaces. Consider a family of games \( \bar{\Gamma}(\bar{\mathcal{T}}, \bar{\mathcal{A}}) \) comprising games \( G(N, \bar{T}, \bar{A}, \tau, \{u_i\}) \) where \( \bar{\mathcal{T}} \) and \( \bar{\mathcal{A}} \) are finite type and action spaces, and \( \bar{T} \) and \( \bar{A} \) are subsets of \( \mathcal{T} \) and \( \mathcal{A} \), respectively. Suppose that \( \bar{\Gamma}(\bar{\mathcal{T}}, \bar{\mathcal{A}}) \) satisfies LC2.

Our first result is that, given any strategy profile, the realized payoff of any individual player \( i \) from playing any strategy is “close” to the ex-ante expected payoff from playing that strategy, with high probability, if the number of players is large enough. Recall some notation: For any player \( i \) and character \( c_i = (a_i, t_i) \), we defined the
function $u^c_i(c_{-i}) := u_i(c_i, c_{-i})$. Further, we denote player $i$’s expected payoff given her type-action character by $\mu^c_i$, i.e., $\mu^c_i = \mathbb{E}[u^c_i(c_{-i})]$.

**Proposition 1 (Inequality for Deviations from Expected Payoff).** Consider a family of finite Bayesian games, $\tilde{\Gamma}$, that satisfies LC2. There exists $\lambda > 0$ such that, for any strategy profile $\sigma$ of $G(N, \tilde{T}, \tilde{A}, \tau, \{u_i\}) \in \tilde{\Gamma}$, the following is true:

For all $i, c_i$, $Pr_{\sigma_{-i}}(|u_i(c) - \mu^c_i| > \alpha) \leq 2e^{-\frac{2(\lambda^2 N - 1)\alpha^2}{N^2}}$.

**Proof.** We prove this proposition using an existing result called the Independent Bounded Differences inequality, which states that the probability that a function of independent random variables deviates from its mean by any quantity $\alpha$ is inversely proportional to the maximum impact that each random variable has on the value of the function. We present this result as a lemma below.

**Lemma 2 (McDiarmid’s Independent Bounded Differences Inequality).** Let $X = (X_1, X_2, \ldots, X_M)$ be a vector of independent random variables with $X_k$ taking values in a set $\mathcal{X}_k$ for each $k$. Suppose that there exist constants $l_k$ ($k = 1, \ldots, M$) such that the real-valued function $g$ defined on $\prod \mathcal{X}_k$ satisfies

$$|g(x) - g(x')| \leq l_k$$

whenever $x$ and $x'$ differ only in the $k^{th}$ coordinate.

Let $\mu$ be the expected value of the random variable $g(x)$. Then, for any $\alpha \geq 0$,

$$Pr(g(x) - \mu \geq \alpha) \leq e^{-\frac{2\alpha^2}{\sum_{k=1}^{M} l_k^2}}.$$  

This result and its proof can be found in McDiarmid (1989). We can apply McDiarmid’s Independent Bounded Differences Inequality to establish Proposition 1. To see how, first note that by Lemma 1, we know that the family of games satisfies the $\lambda$-limited individual impact condition–i.e., we can find a constant $\lambda$ such that

for all $i$, for all $c_{-i} \in \mathcal{C}_{-i}$, whenever $c_{-i}, c'_{-i}$ differ only in one coordinate,

$$|u_i(c_i, c_{-i}) - u_i(c_i, c'_{-i})| \leq \frac{\lambda}{N-1}.$$ 

Fix $\alpha > 0$ and any $c_i$. Applying Lemma 2 to the function $u^c_i$, we get the result. \qed

Proposition 1 makes transparent the role of condition $LC2$ (and the limited in-
individual impact property). The limited individual impact property means that any player’s impact on a rival’s payoffs is bounded and, moreover, decreases with $N$. This implies, in turn, that the probability that a player’s realized payoff deviates from the mean by more than a specified amount vanishes exponentially fast with $N$. Therefore, the ex-post realized payoff from any action is very close to the ex-ante expected payoff from that action. We can use Proposition 1 to establish hindsight stability of Bayesian equilibria for the special case of finite games.

We present such a result below: Specifically, we establish that in a family of finite games satisfying LC2, a special class of approximate equilibria, that we call “strong approximate equilibria,” is approximately hindsight stable if the number of players is large enough. In particular, this implies that any exact Bayesian equilibrium is also approximately hindsight stable if the number of players is large enough. To the best of our knowledge, this is the first robustness result in an environment without anonymity. We will use these results later to prove hindsight stability of equilibria in the general case of infinite type and action spaces.

Before, we present the formal result, we define below the class of approximate equilibria in finite games. An approximate equilibrium ($\varepsilon$-equilibrium) was defined in Section 2. Below, we define (for finite games), a strong $\varepsilon$-equilibrium.

**Definition 10 (Strong $\varepsilon$-Equilibrium).** For $\varepsilon \geq 0$, a strategy profile $\sigma^*$ is a strong $\varepsilon$-equilibrium if for each player $i$, for any type of player $t_i$ that is realized with positive probability, if pure action $a_i$ is played with positive probability under $\sigma^*_i$, then $a_i$ must be an $\varepsilon$-best response to $\sigma^*_{-i}$, given $t_i$.

Note that if $\varepsilon = 0$, this is just the definition of an (exact) equilibrium. It is also worth highlighting that, for $\varepsilon > 0$, if a strategy profile $\sigma^*$ is an $\varepsilon$-equilibrium that prescribes mixed actions for some player $i$, it is not necessary that each of the actions in the support of player $i$’s equilibrium strategy be an approximate best response. This is an important difference between an approximate equilibrium and an exact one. We have the following result about strong approximate equilibria in finite games.

**Theorem 2 (Hindsight Stability of Strong Approximate Equilibria in Finite Games).** Consider a family of finite Bayesian games $\bar{\Gamma}(\bar{\mathcal{T}}, \bar{\mathcal{A}})$ that satisfies LC2. Given $\varepsilon > 0$, and $\eta \geq 0$, there exist positive constants $\bar{\alpha} = \bar{\alpha}(\bar{\Gamma})$, $\bar{\beta} = \bar{\beta}(\bar{\Gamma}, \varepsilon, \eta) < 1$ such that, if $\sigma^*$ is a strong $\eta$-equilibrium of any $N$-player game $G \in \bar{\Gamma}$, then $\sigma^*$ is $(2\eta + \varepsilon, \bar{\alpha}\bar{\beta}^N)$ hindsight-stable.

15We show that LC2 and uniformly bounded strategy spaces are sufficient conditions for hindsight stability in finite games. We leave unanswered the question of what a parsimonious necessary condition is. Clearly, some form of “continuity” will be necessary for hindsight stability.
Proof. Let $\varepsilon > 0$ and $\eta \geq 0$ be given. Suppose that $\sigma^*$ is an arbitrary strong $\eta$-equilibrium of an $N$-player game $G$ in $\Gamma(\hat{F}, s\hat{F})$ satisfying LC2. Fix any player $i$ and a type-action character $c_i = (a_i, t_i)$, that is realized with positive probability in equilibrium (i.e. type $t_i$ has a positive probability of being realized, and $a_i$ is played with positive probability in equilibrium by type $t_i$). Since the type space is finite here, we need only consider positive probability types. Define $D_{i,c_i} = \{c_{-i} : u_i(\tilde{c}_i, c_{-i}) - u_i(c) > 2\eta + \varepsilon$ for some $\tilde{c}_i$ with $\tilde{t}_i = t_i\}$. In other words, $D_{i,c_i}$ denotes the rival type-action character profiles where player $i$ with character $c_i = (a_i, t_i)$ can unilaterally change her action choice and gain more than $2\eta + \varepsilon$. Let $D_i$ denote $\bigcup c_i D_{i,c_i}$, and let $D = \bigcup_i D_i$. We will show that, as $N$ becomes large, the probability of $D$ goes to zero. Now,

$$u_i(\tilde{c}_i(c_{-i}) - u_i(c_{-i}) = \left(\mu_i(c_{-i}) - \mu_{\tilde{c}_i}(c_{-i})\right) + \left(\mu_{\tilde{c}_i}(c_{-i}) - \mu_{\tilde{c}_i}(c_{-i})\right).$$

If the first and last terms are both less than $\frac{\eta + \varepsilon}{2}$, and the second term is less than $\eta$, then the left hand side is less than $2\eta + \varepsilon$.

We know, by the limited individual impact property and Proposition 1, for any characters $c_i$ and $\tilde{c}_i$, we have

$$Pr_{\sigma_{-i}}\left(|u_i(\tilde{c}_i(c_{-i}) - \mu_{\tilde{c}_i})| \leq \frac{\eta + \varepsilon}{2}\right) > 1 - 2e^{-\frac{(\eta + \varepsilon)^2(N-1)}{x^2}}, \tag{1}$$

and similarly,

$$Pr_{\sigma_{-i}}\left(|u_i(c_{-i}) - \mu_{\tilde{c}_i}| \leq \frac{\eta + \varepsilon}{2}\right) > 1 - 2e^{-\frac{(\eta + \varepsilon)^2(N-1)}{x^2}}. \tag{2}$$

Further, since $\sigma$ is a strong $\eta$-equilibrium, we know that any action chosen with positive probability must be approximately optimal, given the type, i.e., $\mu_{\tilde{c}_i} - \mu_{c_i} \leq \eta$. From the complements of inequalities (1) and (2), it follows that,

$$Pr_{\sigma_{-i}}\left(u_i(\tilde{c}_i(c_{-i}) - u_i(c_{-i}) > 2\eta + \varepsilon\right) \leq 4e^{-\frac{(\eta + \varepsilon)^2(N-1)}{x^2}}.$$

Therefore, given a type of player $i$ and a corresponding action that is realized in equilibrium, the probability that any deviation yields more than an $2\eta + \varepsilon$ gain for player $i$ is, at most, $4|C|e^{-\frac{(\eta + \varepsilon)^2(N-1)}{x^2}}$, where $|C|$ is the total number of possible type-
action character profiles—i.e., we have $Pr_{\sigma'_{-i}}(D_i) \leq 4|C|e^{-\frac{(\eta + \varepsilon)^2}{2}(N-1)}$. Therefore, $Pr(D)$, the probability that any player can profit by more than $2\eta + \varepsilon$ by deviating is, at most, $4N|C|e^{-\frac{(\eta + \varepsilon)^2}{2}(N-1)}$. Equivalently, we have shown that the strategies are $(2\eta + \varepsilon, 4N\alpha'\beta'^{N-1})$ hindsight-stable. So, we have found constants $\alpha' = |C|$ and $\beta' = e^{-\frac{(\eta + \varepsilon)^2}{2N}} < 1$ such that all equilibria are $(2\eta + \varepsilon, 4N\alpha'\beta'^{N-1})$ hindsight-stable. Note that we can replace constants $\alpha'$ and $\beta'$ by bigger constants $\bar{\alpha}$ and $\bar{\beta}$ such that $N\alpha'\beta'^{N-1} < \bar{\alpha}\bar{\beta}^N$ for all $N$. This completes the proof. \hfill \Box

Note that the above result immediately implies that any exact equilibrium of a family of finite games $\hat{\Gamma}(\hat{T}, \hat{A})$ that satisfies LC2, is also asymptotically hindsight stable. This is simply the special case of $\eta = 0$.

### 3.2 Finite Grid Reductions

Now, we return to the case of games with infinite type and action spaces. We will use our results on hindsight stability in finite games to establish hindsight stability of equilibria in the infinite case. Consider a Bayesian game $G(N, T, A, \tau, \{u_i\})$ in which the type and action spaces $T$ and $A$ are compact subsets of uniformly bounded type and action spaces $\mathcal{T}$ and $\mathcal{A}$. Fix $\Delta > 0$ small. We define a new game $\hat{G}$ that we call the $\Delta$-finite grid reduction. We need some additional definitions and notation.

For every $x \in \mathbb{R}$, define $r(x) := \hat{x} = \Delta \max\{k : k \in \mathbb{Z} \text{ such that } k\Delta \leq x\}$. For any $x \in \mathbb{R}^m$, define $r(x) := \hat{x} = (\hat{x}_1, \ldots, \hat{x}_m)$.

Associated with the compact type space $T$, we can define a finite type-space $\hat{T}_i = \{r(t_i) : t_i \in T_i\}$, and $\hat{T} = \prod \hat{T}_i$. Similarly, associated with the compact action space $A$, we define a finite action space $\hat{A}_i = \{r(a_i) : a_i \in A_i\}$, and $\hat{A} = \prod \hat{A}_i$. Define the corresponding type-action character spaces $\hat{C}_i = \hat{T}_i \times \hat{A}_i$ and $\hat{C} = \prod \hat{C}_i$. Similarly, we can also define $\hat{\mathcal{T}}$ and $\hat{\mathcal{A}}$.

Associated with the prior marginal distributions of types $\tau_i$, define distributions $\hat{\tau}_i$ over $\hat{T}_i$ such that $\hat{\tau}_i(\hat{t}_i) = \tau_i(r^{-1}(\hat{t}_i))$. As usual, $\hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_n)$. For any character profile $\hat{c}$, define player $i$’s payoff $\hat{u}_i(\hat{c}) = \frac{1}{2} \left[ \sup_{c \in r^{-1}(\hat{c})} u_i(c) + \inf_{c \in r^{-1}(\hat{c})} u_i(c) \right]$. As before, let $\hat{U}_i(\sigma)$ denote the expected payoff of player $i$ from strategy profile $\sigma$.

We call the new game defined by $\hat{G} = (N, \hat{T}, \hat{A}, \hat{\tau}, \hat{u})$ the finite $\Delta$-grid reduction of the original game $G = (N, T, A, \tau, u)$. Compactness implies that $\hat{G}$ is a game with

\footnote{Notice that all parameters on the right hand side of this inequality are independent of the selected equilibrium $\sigma^*$. Thus this bound holds uniformly for any selected strong $\eta$-equilibrium.}
finite type and action spaces. For any strategy $\sigma_i$ in the original game $G$, we can define an associated strategy $\hat{\sigma}_i$ in the $\Delta$-finite reduction $\hat{G}$ by $\hat{\sigma}_i(\hat{c}_i) = \sigma_i(r^{-1}(\hat{c}_i))$. For a strategy profile $\sigma$ of $G$, $\hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_n)$. Below, we establish some results about the relationship between a game $G$ and its finite grid reduction $\hat{G}$.

**Proposition 2.** Consider a family of games $\Gamma(\mathcal{F}, \mathcal{A})$, in which the strategy space is uniformly bounded, (with bounds $k_T$ and $k_A$ on the dimensionality of $\mathcal{F}$ and $\mathcal{A}$, respectively), and payoff functions satisfy LC1 and LC2 (with constants $K$ and $L$, respectively). Let $G(N, T, A, \tau, \{u_i\})$ be a Bayesian game from the family $\Gamma(\mathcal{F}, \mathcal{A})$, and let $\hat{G}$ be a $\Delta$-finite reduction of $G$. Then

(A) For every equilibrium $\sigma$ of $G$, the associated strategy $\hat{\sigma}$ is a strong $2(k_T + k_A)(K + L)\Delta$-equilibrium of $\hat{G}$.

(B) If $\hat{\sigma}$ is a $(\eta, \rho)$ hindsight-stable equilibrium of $\hat{G}$, then any associated strategy $\sigma$ of the original game $G$ is $(\eta + (k_T + k_A)(K + 2L)\Delta, \rho)$ hindsight-stable in $G$.

**Proof.** Proof of (A): Fix $\Delta > 0$ small. We first show that, for any strategy profile $\sigma$ of $G$ and the associated strategy profile $\hat{\sigma}$ in the $\Delta$-finite grid reduction, $\hat{G}$, and for any positive probability type $\hat{t}_i$ of player $i$, the following is true:

$$|\hat{U}_i(\hat{\sigma}|\hat{t}_i) - U_i(\sigma|r^{-1}(\hat{t}_i))| \leq (k_T + k_A)(K + L)\Delta. \quad (3)$$

To see why, note that for $\hat{c} \in \hat{C}_i$, and for any $c, c' \in r^{-1}(\hat{c})$, the conditions LC1 and LC2 imply that

$$|u_i(c_1, \ldots, c_n) - u_i(c'_1, \ldots, c'_n)|$$

$$\leq |u_i(c_1, \ldots, c_n) - u_i(c'_1, c_2, \ldots, c_n)| + |u_i(c'_1, c_2, \ldots, c_n) - u_i(c'_1, c'_2, \ldots c_n)| + \ldots + |u_i(c'_1, \ldots, c_n) - u_i(c'_1, c'_2, \ldots c'_n)|$$

$$\leq K(k_T + k_A)\Delta + \frac{L}{N-1}(k_T + k_A)\Delta + \ldots + \frac{L}{N-1}(k_T + k_A)\Delta$$

$$\leq (k_T + k_A)(K + L)\Delta.$$

Then, (3) follows from the definition of $\hat{u}_i$ and $\hat{U}_i$.

We will now use inequality (3) above to establish that $\hat{\sigma}$ is a strong $2(k_T + k_A)(K + L)\Delta$-equilibrium of $\hat{G}$. We do this in two steps. We first show that it is an approximate equilibrium, and then show that in particular, it is a strong approximate equilibrium.

Now, suppose that $\sigma$ is an equilibrium strategy profile of $G$. Let $\hat{\sigma}$ denote the associated strategy in the $\Delta$-finite reduction of $G$. Fix a player $i$ and a type of player $i$ that is realized with positive probability, $\hat{t}_i$. Let $(\hat{\sigma}_i', \hat{\sigma}_{-i})$ denote a strategy profile in the $\Delta$-finite reduction, in which player $i$ unilaterally deviates from $\hat{\sigma}$. Now, we can
where the second inequality follows from the fact that \( \sigma \) is an equilibrium of \( G \). This establishes that \( \hat{\sigma} \) is a \( 2(k_T + k_A)(K + L) \Delta \)-equilibrium of \( \hat{G} \).

Next we show, that in particular, \( \hat{\sigma} \) is also a strong approximate equilibrium of \( \hat{G} \). The intuition is that since the strategy profile \( \hat{\sigma} \) is derived from an exact equilibrium \( \sigma \) of \( G \), any action that is played with positive probability under \( \hat{\sigma} \) is “close” to an action that is played with positive probability under \( \sigma \), and therefore the payoff from any such action must be approximately optimal. We show this formally below:

Suppose that pure action \( \hat{a}_i \) is played with positive probability under \( \hat{\sigma} \). To prove that \( \hat{\sigma} \) is a strong approximate equilibrium, we need to show that the pure action \( \hat{a}_i \) is an approximate best response to \( \hat{\sigma}_{-i} \) given \( \hat{\sigma}_i \). Consider a potential deviation by player \( i \) given type \( \hat{t}_i \) to an alternate action \( \hat{a}_i \). By construction of the \( \Delta \)-finite reduction and the associated strategy \( \hat{\sigma} \), we know that, given \( r^{-1}(\hat{t}_i) \), there exists action \( a_i \in r^{-1}(\hat{a}_{-i}) \) that is played with positive probability in the equilibrium \( \sigma \) of \( G \), and we have

\[
\hat{U}_i(a_i, \hat{\sigma}_{-i}|\hat{t}_i) - \hat{U}_i(\hat{a}_i, \hat{\sigma}_{-i}|\hat{t}_i) \\
\leq U_i(a_i, \sigma_{-i}|r^{-1}(\hat{t}_i)) + (k_T + k_A)(K + L)\Delta - U_i(\hat{a}_i, \hat{\sigma}_{-i}|r^{-1}(\hat{t}_i)) + (k_T + k_A)(K + L)\Delta \\
\leq 2(k_T + k_A)(K + L)\Delta.
\]

Note that the first inequality follows from the derivation of (3). The second inequality follows from the fact that \( \sigma \) is an exact equilibrium of \( G \), which implies that any action \( a_i \) that is played with positive probability must be an exact best response for player \( i \) given his type.

Proof of (B): Let \( \hat{S} = \{ \hat{c} : \text{no player can gain more than } \eta \text{ by deviating from } \hat{c} \} \). Let \( S = r^{-1}(\hat{S}) \). By definition of the finite grid reduction, \( Pr_\eta(\hat{S}) = Pr_\sigma(S) \). Further, since \( \hat{\sigma} \) is \((\eta, \rho)\) hindsight-stable, we know that \( Pr_\eta(\hat{S}) \geq 1 - \rho \). Now, if a player cannot gain by more than \( \eta \) by deviating at \( \hat{c} \), then at any \( c \in r^{-1}(\hat{c}) \), she cannot improve by more than \( \eta + (k_T + k_A)(K + 2L)\Delta \). This follows from the proof of (3). Therefore, the probability that some player can deviate and make a gain of more than
\[ \eta + (k_T + k_A)(K + 2L) \Delta \text{ must be less than } \rho. \]

3.3 Establishing Hindsight Stability in the General Case

We are now equipped to prove Theorem 1 - the central result of this paper that establishes hindsight stability of equilibria on games with infinite type and action spaces. Consider a family of games \( \Gamma(\mathcal{T}, \mathcal{A}) \) in which the strategy space is uniformly bounded, and payoff functions satisfy LC1 and LC2 with Lipschitz constants \( K \) and \( L \), respectively. Fix \( \varepsilon > 0 \), small. Consider \( \Delta = \varepsilon \left[ \frac{K + L}{2(k_T + k_A)(5K + 6L)} \right] \). For this \( \Delta \), we can derive the corresponding family of \( \Delta \)-finite reductions of games in \( \Gamma \), and denote it by \( \bar{\Gamma} \).

Consider an arbitrary equilibrium \( \sigma^* \) of a game \( G \) in \( \Gamma \), and let \( \hat{G} \in \bar{\Gamma} \) denote the \( \Delta \)-finite reduction of \( G \). We know from Proposition 2, that the associated equilibrium \( \hat{\sigma}^* \) of \( \hat{G} \) is a strong \( \left( \frac{\varepsilon(K+L)}{5K + 6L} \right) \)-equilibrium of \( \hat{G} \). Substituting for \( \Delta \), we have that \( \hat{\sigma}^* \) is a strong \( \varepsilon \left( \frac{K + L}{5K + 6L} \right) \)-equilibrium of \( \hat{G} \). We can then apply our result about hindsight stability of strong approximate equilibria in finite games (Theorem 2) to \( \bar{\Gamma} \).

Theorem 2 implies (for \( \frac{1}{\varepsilon} > 0 \)) that there are constants \( \bar{\alpha} \) and \( \bar{\beta} < 1 \) such that \( \hat{\sigma}^* \) is \( \left( \varepsilon, \bar{\alpha} \bar{\beta}^N \right) \) hindsight-stable in \( \hat{G} \). Finally, we apply part (B) of Proposition 2 to show that the original equilibrium \( \sigma^* \) of a game \( G \in \Gamma \) is \( \varepsilon \left( \frac{K + L}{5K + 6L} \right) \) hindsight-stable. Note that the value of the constants \( \bar{\alpha} \) and \( \bar{\beta} \) depend on the particular finite grid reduction and, therefore, depend both on the family of games \( \Gamma \) and on \( \varepsilon \). In other words, given the family \( \Gamma \) and \( \varepsilon \), we can find constants \( \alpha := \alpha(\Gamma, \varepsilon) \) and \( \beta := \beta(\Gamma, \varepsilon) < 1 \), such that any equilibrium \( \sigma^* \) of a game \( G \in \Gamma \) is \( \varepsilon, \alpha \beta^{N-1} \) hindsight-stable. This establishes the main theorem.

4 Concluding Remarks

Little was known about the stability properties of Bayesian equilibria in games with infinite types and actions, or in games that are not anonymous. In this paper, we study a class of Bayesian games in which the type and action spaces are infinite and players are not anonymous. We impose two regularity conditions on the payoff functions—variants of Lipschitz Continuity—and show that this is enough to guarantee hindsight stability of equilibria if the number of players is large. Notice that these are sufficient conditions for hindsight stability. It would be interesting to investigate if, and in what sense, these conditions are also necessary.\footnote{For instance, we use condition LC1 to extend the robustness result from finite games to infinite games, using the finite grid approximation method. It may be interesting to ask if there is a more direct proof that}
A promising line of research may be to study other ways of modifying the notions of stability and relaxing the restrictions on the family of games. For instance, we might consider an alternate notion of hindsight stability. The notion of hindsight stability in this paper requires that no player have an incentive to revise her action after learning all the information about the opponents’ types and actions. For full hindsight stability, this means also that players would not have an incentive to revise their choices after learning partial information about their opponent’s types and actions. However, as illustrated by an example in Kalai (2004), this is not the case for approximate \((\varepsilon, \rho)\)-hindsight-stability. Even if an equilibrium is \((\varepsilon, \rho)\)-hindsight-stable in the sense of this paper, there may still be partial information that reaches a player ex-post with probability greater than \(\rho\), and based on this information, the player could increase her payoff by more than \(\varepsilon\) by a unilateral ex-post deviation. Strong \((\varepsilon, \rho)\)-hindsight-stability of an equilibrium (see Kalai (2004)) requires that there be no significant probability for ex-post unilateral revisions that lead to meaningful gains by individual players even after they obtain partial hindsight information about the play of the game. It is not clear whether this stronger condition holds for the family of games presented in this paper, or if other stronger properties such as structural robustness hold. We leave these issues out of the current paper, as addressing them would require the formulation of extensive versions of the game, which is significantly more difficult with a continuum of types and actions. It is also not clear if weaker robustness properties of Bayesian equilibrium would hold if types were not independent or if we had payoff functions that displayed discontinuities (see Gradwohl and Reingold (2010), for example). This line of investigation would be particularly important for applications.

__perhaps does not use LC1, but some alternate weaker condition.__
References


