Cascading Failures in Production Networks

David Rezza Baqaee†

Latest draft available at: http://tinyurl.com/ltdu2wr

January 15, 2015

Abstract

I show how the extensive margin of firm entry and exit can greatly amplify idiosyncratic shocks in an economy with a production network. I show that canonical input-output models, which lack the extensive margin of firm entry and exit, have some crucial limitations. In these models, the systemic importance of a firm does not respond to productivity shocks, depends only on the firm’s role as a supplier, and is equal to or well-approximated by the firm’s size. This means that for every canonical input-output model, there exists a non-interconnected model that has the same aggregate response to productivity shocks. I show that when we allow for entry and exit, the systemic importance of a firm responds endogenously to productivity shocks, depends on a firm’s role not just as a supplier but also as a consumer, and a firm’s systemic influence is no longer well-approximated by its size. Furthermore, I show that non-divisibilities in systemically important industries can cause one failure to snowball into a large-scale avalanche of failures. In this sense, shocks can be amplified as they travel through the network, whereas in canonical input-output models they cannot.

First Version: September 2013. I thank John Campbell, Emmanuel Farhi, Ben Golub, Marc Melitz, and Alp Simsek for their guidance. I thank Daron Acemoglu, Pol Antras, Natalie Bau, Aubrey Clark, Gita Gopinath, Alireza Tahbaz-Salehi, Thomas Steinke, Oren Ziv, and David Yang for helpful comments. I thank Ludvig Sinander for his detailed comments on an earlier draft.

†Department of Economics, Harvard University, Cambridge, MA, 02138, baqaee@fas.harvard.edu.
1 Introduction

In this paper I show how the extensive margin of firm entry and exit can dramatically alter the properties of macroeconomic models with production networks. I model cascades of failures among firms linked through a production network, and show how the network propagates and amplifies shocks through supply and demand chains. This paper contributes to the literature on the microeconomic sources of aggregate business cycle fluctuations.

In a recent paper Acemoglu et al. (2012) relate the following anecdote, which illustrates the basic mechanism that I model and demonstrates its real-world relevance:

In the fall of 2008, rather than asking for government assistance for Ford, Alan R. Mulally, the chief executive of Ford Motor Co., requested that the government supports General Motors and Chrysler. His reasoning for asking government support for his company’s traditional rivals was that the failure of either GM or Chrysler would lead to the potential failure of their suppliers, and because Ford depended on many of the same suppliers as the other two automakers, it would also find itself in perilous territory.

The scenario Mulally described, which was averted through government intervention in the United States, has now come to pass in Australia. In May 2013, Ford Australia announced that they would stop manufacturing cars in 2017. Seven months later, GM Australia announced they would also stop manufacturing cars in 2017. Three months after that, in February 2014, Toyota Australia also announced that it would close its manufacturing plants at the same time. This effectively ended automobile manufacturing in Australia. The Australian government predicts that this will result in the loss of over 30,000 jobs, a figure they arrived at by adding the number of people directly employed by the three automakers and the Australian car parts industry.

While these examples demonstrate that firm exit (and entry) can have important spill-overs on other firms, standard macroeconomic models do not allow for this possibility. In this paper, I explicitly incorporate the extensive margin of firm entry into an input-output model of production. I show how the extensive margin alters the quantitative and qualitative properties of the model. First, I show that the standard input-output macroeconomic models that follow Long and Plosser (1983), like Acemoglu et al. (2012), Atalay (2013), or Baqee (2014) have the property that their responses to productivity shocks can be summarized in terms of a few exogenous sufficient statistics. Once we compute the relevant sufficient statistics, which are closely related to the equilibrium size of firms, we can discard the network structure. In other words, I show that there are disconnected economies, with different structural parameters (and sometimes exogenous wedges), that behave precisely like the network models. This means that, without the extensive margin, it is not the interconnections per se, but how those interconnections affect a firm’s size that determines a firm’s systemic importance. If we can arrive at the same sufficient statistics using
a different (perhaps degenerate) network-structure, the equilibrium responses will be the same. This fact explains why the theoretical implications of the granular hypothesis of Gabaix (2011), where business cycles are driven by large firms, are observationally equivalent to the theoretical implications of the network hypothesis of Acemoglu et al. (2012), where business cycles are driven by well-connected firms. Furthermore, in canonical input-output models, systemic importance depends only on a firm’s role as a supplier. As long as firms $i$ and $j$ have the same strength connections to the same customers, then their systemic influence will be the same, regardless of what $i$ and $j$’s supply chains look like.

When we allow for entry and exit, the sufficient-statistic approach breaks down. This is because the extensive margin makes “systemic importance” an endogenously determined value that is not well-approximated by equilibrium size or prices. A firm that may seem like a small player, when measured by sales, can have potentially large impacts on aggregate outcomes. On the other hand, a firm that may seem like a key player, as measured by sales, can have relatively minor effects on the equilibrium. Furthermore, the endogenously-determined measures of systemic influence depend on a firm’s role as both a supplier and as a consumer, as well as on how many close substitutes there are for the firm.

This allows us to combine the granular hypothesis of Gabaix (2011) and the network hypothesis of Acemoglu et al. (2012) in a new and interesting way. In particular, I show that once firms have positive mass, extensive margin shocks can be locally amplified via interconnections. In other words, when a large and well-connected firm exits, it can set off an avalanche of firm failures that actually gets larger as it gathers steam. This type of amplification is not possible unless we have both granularity and network connections. With positive mass, the model satisfies the criteria of Scheinkman and Woodford (1994) for self-organized criticality: it exhibits strong local interactions that are significantly nonlinear.

This paper also contributes to the wider literature on diffusion on social networks, by bridging the gap between two alternative modelling traditions. Loosely speaking, there are two popular approaches to modelling diffusion on social networks. First, there are continuous input-output type models like Acemoglu et al. (2012). Here, nodes influence each other in continuous ways – shocks travel away from their source like waves and slowly die out. The strength of the connections between the nodes controls the rate of decay. Such shocks, sometimes called pulse processes, are characterized by geometric sums. I show that these models are incapable of local amplification: a productivity shock to an industry will always have its largest effect at its source, and the shock decays as it travels through the connections. These structures were first studied by Leontief (1936) in his input-output model of the economy.

The other camp consists of models that behave discontinuously, as typified by Morris (2000) or Elliott et al. (2012). In this class of models, sometimes called threshold models, each node has a threshold and is either active or inactive. When a node crosses its threshold it changes states and,
by changing states, pushes its neighbors closer to their thresholds. Such models are frequently used to study the spread of epidemics, products, or even ideas. One of the earliest and most influential threshold models is the Schelling (1971) model of segregation. Threshold models do not have wave-like properties since the rate at which shocks decay are not geometric. Crucially, these models are capable of generating local amplification – that is, shocks can be amplified as they travel through the network; however these models are notoriously difficult to analyze.

In this paper, I consider a model that bridges the gap between the continuous and discrete models. Specifically, I explicitly account for the mass of firms in a given industry. Industries with a continuum of firms behave continuously – the mass of firms responds continuously to shocks. On the other hand, lumpy industries, with only a few firms, behave discontinuously. For instance, a negative shock to an unconcentrated industry, say hairdressing, will result in some fraction of hairdressers exiting. The fraction exiting will be a continuous function of the size of the shock. The effect on a neighboring industry will be attenuated by the strength of its connections to hairdressing. However, a negative shock to a highly concentrated industry, like automobile manufacturing, will have no effect on the number of firms unless it is large enough to force an exit. But once a large firm exits, it imparts an additional impulse to the size of the shock which can trigger a cascade. Because the model is flexible enough to express both behaviors, I can provide conditions under which we can expect a continuous approximation to a discontinuous model to perform badly.

The idea of cascading – domino-like – chain reactions also appears outside of economics. In particular, models of contagion and diffusion like the threshold models considered by Kempe et al. (2003) have these features. In these models, notions of connectedness play a key role since the only way contagion can spread is via connections between nodes. An interesting implication of embedding a contagion model into a general equilibrium economy is the role prices and aggregate demand play – a role that does not have analogues in other threshold models. Typically, in a threshold model, shocks can only travel along edges. Contagion can only spread to nodes who are connected to an infected node. This important intuition breaks down in general equilibrium models since all firms are linked together via aggregate demand. This means that general equilibrium forces can act like long-distance carriers of disease. Shocks in one fragile industry, like the financial industry, can jump via aggregate demand, to a different fragile industry like automobile manufacturing even if these two industries are not connected.

The structure of paper is as follows. In section 2, I set up the model and define its equilibrium. In section 3, I characterize the equilibrium conditional on the mass of entrants in each industry and define some key centrality measures. In section 4, I study how the model behaves when the extensive margin of firm entry and exit is shut down. I prove results showing that the network structure can be summarized by sufficient statistics related to size. I also show that we can think of these models as non-interconnected models with different parameters. Finally, I show that this
class of models is incapable of local amplification of shocks. In section 5, I allow firm entry and exit. First, I characterize the model’s responses to shocks in the limit where all firms are massless. I show that sufficient statistics are no longer available and that systemic importance is endogenous. Then, I consider the case when firms can have positive mass, and show that with atomistic firms, shocks can be locally amplified. I prove an inapproximability result showing conditions under which we should expect a continuous approximation to a discontinuous model to perform badly. In section 6, motivated by the Ford example I discuss above, I consider conditions under which firms’ incentives align with those of society. Specifically, when can we trust one firm’s testimony about whether or not another firm should be bailed out. I conclude in section 7.

2 Model

In this section, I spell out the structure of the model and define the equilibrium. There are three types of agents: households, firms, and a government. Each firm belongs to an industry, and there are $N$ industries.

The households in the model are homogenous with a unit mass. The representative household maximizes utility

$$U(c_1, \ldots, c_N) = \left( \sum_{k=1}^{N} \beta_k \frac{c_k}{\sigma_k} \right)^{\sigma_k + 1},$$

where $c_k$ represents composite consumption of varieties from industry $k$ and $\sigma > 0$ is the elasticity of substitution across industries. The composite consumption good produced by industry $k$ is given by

$$c_k = \left( \sum_{i=1}^{N_k} \Delta_k c(k, i)^{\epsilon_k - 1} \right)^{\epsilon_k},$$

where $c(k, i)$ is household consumption from firm $i$ in industry $k$ and $\epsilon_k > 1$ is the elasticity of substitution across firms within industry $k$. Here, $N_k$ is the number of firms active in industry $k$ and $\Delta_k$ is the mass of each firm. The assumption that $\Delta_k$ is constant for all firms in industry $k$ means firms in each industry are homogenous. The total mass of firms in industry $k$ is given by $M_k = N_k \Delta_k$. The household’s budget is given by

$$\sum_{k,j} p(k, i)c(k, i) = w l + \sum_{k,i} \pi(k, i) - \tau,$$

where $p(k, i)$ is the price of firm $i$ in industry $k$ and $\pi(k, i)$ is firm $i$ in industry $k$’s profits. The wage is $w$ and labor is inelastically supplied at $l$. For the rest of the paper, and without loss of generality, we take labor to be the numeraire so that $w = 1$, and fix the supply of labor $l = 1$. Lump sum taxes by the government are denoted $\tau$. 

Let firm $i$ in industry $k$ maximize profits

$$\pi(k, i) = p(k, i)y(k, i) - \sum_{l=1}^{N} \sum_{j} p(l, j)x(k, i, l, j) - w_l(k, i) - w_f k + \tau_k,$$

where $p(k, i)$ is the price and $y(k, i)$ is the output of the firm. Inputs from firm $j$ in industry $l$ are $x(k, i, l, j)$ and labor inputs are $l(k, i)$. Finally, in order to operate, each firm pays a fixed cost $f_k$ in units of labor and the firm potentially receives a lump sum subsidy $\tau_k$. The mass parameter $\Delta_k$ controls how finely the fixed costs of industry $k$ can be split up – in other words, it captures increasing returns to scale at the industry level. The firm’s production function (once the fixed cost has been paid) is constant returns to scale

$$y(k, i) = \left( \frac{1}{\alpha_k} (z_k l(k, i))^{\frac{\omega_k}{\sigma}} + \sum_{l=1}^{N} \omega_{kl} x(k, i, l) \right)^{\frac{1}{\sigma}}.$$

Here, $\sigma > 0$ is again the elasticity of substitution among inputs, and $\omega_{kl}$ is the CES share parameter for how intensively firms in industry $k$ use composite inputs from industry $l$. The $N \times N$ matrix of $\omega_{kl}$ determine the network-structure of this economy. One can think of this matrix as the adjacency matrix of a weighted directed graph. The parameter $\alpha_k > 0$ gives the intensity with which firms in industry $k$ use labor.

Labor productivity shocks, like the ones considered by Acemoglu et al. (2012) and Atalay (2013) are denoted by $z_k$. Note that when $\sigma \neq 1$, a productivity shock $z_k$ to industry $k$ is equivalent to changing that industry’s labor intensity from $\alpha_k$ to $\alpha_k z_k^{\sigma-1}$. Therefore, as long as $\sigma \neq 1$, we can think of $\alpha_k$ as including both the productivity shock and the labor intensity. This way we do not need to directly make reference to the shocks $z$ since they are just equivalent to changing $\alpha$. This equivalence breaks down when $\sigma = 1$, and in those cases, we shall have to work directly with $z$. For the majority of this paper, I focus on the propagation of productivity shocks. When the elasticity of substitution is equal to one, these are precisely the shocks considered by Acemoglu et al. (2012). The same methods can easily be used to study other shocks. I defer the discussion of how other shocks, like fixed-cost shocks or demand shocks, would affect the results to the end of the paper.

The composite intermediate input from industry $l$ used by firm $i$ in industry $k$ is

$$x(k, i, l) = \left( \sum_{j=1}^{N_i} \Delta_{jl} x(k, i, l, j) \right)^{\frac{\epsilon_l}{\tau_l}},$$

where $\epsilon_l$ is the elasticity of substitution across different firms within industry $l$. Note that the elasticities of substitution are the same for all users of an industry’s output.
The government runs a balanced budget so that
\[ \sum_k \tau_k = \tau. \]

We study the subgame perfect Nash equilibrium. In period 1, entry decisions are made simultaneously. In period 2, firms play monopolistic competition conditional on period 1’s entry decisions.

**Definition 2.1.** A monopolistically competitive equilibrium is a collection of prices \( p(i,k) \), wage \( w \), and input demands \( x(i,k,l,j) \), outputs \( y(i,k) \), consumptions \( c(i,k) \) and labor demands \( l(i,k) \) such that for mass of entrants \( \{M_k\}_{k=1}^N \) and vector of productivity shocks \( z_k \),

(i) Each firm maximizes its profits taking as given the industrial price level and industrial demand,

(ii) the representative household chooses consumption to maximize utility,

(iii) the government runs a balanced budget,

(iv) markets for each good and labor clear.

Note that the productivity shock is known at the start of the game. Changing the information structure to study the effects of uncertainty on the actions of agents is an interesting extension that I leave for future work.

Let \( \Pi : \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R}_+^N \) be the function mapping the masses of entrants \( M \) and vector of productivity shocks \( z \) to industrial profits assuming monopolistic competition in period 2. In theorem 3.3, I analytically characterize this function.

**Definition 2.2.** A vector of integers \( \{N_k\}_{k=1}^N \) is an equilibrium number of entrants if

\[ \Pi_i(M_i,M_{-i},z) \geq 0 > \Pi_i(M_i + \Delta_i,M_{-i},z) \quad (i \in \{1,2,\ldots,N\}), \]

where \( M_j = N_j \Delta_j \), for all \( j \).

Intuitively, a vector of integers is an equilibrium number of entrants if all firms make non-negative profits, and the entry of an additional firm in any industry results in firms in that industry making negative profits.

**Notation**

Let \( e_i \) denote the \( i \)th standard basis vector. Let \( \Omega \) be the \( N \times N \) matrix whose \( ij \)th element is equal to \( \omega_{ij} \). Let \( \alpha \) and \( \beta \) be the \( N \times 1 \) vectors consisting of \( \alpha_i \)s and \( \beta_i \)s. Let \( \tilde{M} \) be the \( N \times N \) diagonal
matrix whose $i$th diagonal element is equal to $M_i^{-\frac{1}{\epsilon_i-1}}$, and let $M$ be the $N \times N$ diagonal matrix whose $i$th element is $M_i$. Finally, let $\mu$ be the $N \times N$ diagonal matrix whose $i$th diagonal element is the mark-up $\frac{\epsilon_i}{(\epsilon_i - 1)}$ charged by firms in industry $i$. Let $\circ$ denote the element-wise or Hadamard product, and $\text{diag} : \mathbb{R}^N \to \mathbb{R}^{N^2}$ be the operator that maps a vector to a diagonal matrix.

**Definition 2.3.** An economy $E$ is defined by the tuple $E = (\beta, \Omega, \alpha, \epsilon, \sigma, f, \Delta)$. The vector $\beta$ contains household taste parameters, $\Omega$ captures the input-output share parameters, $\alpha$ contains the industrial labor share parameters, $\epsilon > 1$ is the vector of industrial elasticities of substitution, $\sigma > 0$ is the cross-industry elasticity of substitution, $f$ is the vector of fixed costs, and $\Delta$ is the vector of masses of firms in each industry.

Before analyzing the model, it helps to define some key statistics. These are standard definitions from the literature on monopolistic competition. See, for example, Bettendorf and Heijdra (2003).

**Definition 2.4.** The price index for industry $k$ is given by

$$p_k = \left( \sum_{i} N_k p(k, i)^{1-\epsilon_k} \right)^{\frac{1}{1-\epsilon_k}},$$

and the total composite output of industry $k$ is given by

$$y_k = \left( \sum_{i} N_k y(k, i)^{\frac{1}{\epsilon_k}} \right)^{\frac{\epsilon_k}{\epsilon_k - 1}}.$$

The consumer price index, which represents the price level for the household, is given by

$$P_c = \left( \sum_{k} N \beta_k p_k^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

and total consumption by the household is given by

$$C = \left( \sum_{k=1}^{N} \beta_k^{\frac{1}{\epsilon_k}} \frac{\epsilon_k - 1}{\epsilon_k} \right)^{\frac{\epsilon_k}{\epsilon_k - 1}},$$

which is just the utility of the household.

These are the “ideal” price and quantity averages for each industry. The reason we do not simply average prices to get a price index or add outputs to get total output is because even within each industry, each firm is producing a slightly different product. When the elasticity of substitution $\epsilon_k = 0$, then the price index for that industry is simply the sum of all the prices since
there is no substitution and a consumer of this industry must buy all the varieties. When the
elasticity of substitution $\varepsilon_k \to \infty$, the price index for an industry is just the minimum price, since
households will only purchase from the cheapest firm in each industry. An important special case
is when $\varepsilon_k \to 1$, where the price index is a geometric average of the industrial prices.

3 Monopolistic Competition Subgame

In this section, I characterize the equilibrium in the monopolistic competition subgame of period
2, conditional on the number of entrants in each industry. Before stating any results, it helps to
define some key industrial statistics.

Let us define supply-side and demand-side centrality measures.

**Definition 3.1.** The supplier centrality is

$$\tilde{\beta}' = \beta' \Psi_s,$$

where

$$\Psi_s = (I - \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega)^{-1} = \sum_{n=0}^{\infty} \left( \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega \right)^n.$$

This captures the frequency with which each industry appears in supply chains. The $k$th element of $\tilde{\beta}$ captures demand from the household that reaches industry $k$, whether directly or indirectly through other industries who use $k$’s products. The following helps establish why we might care about $\tilde{\beta}$.

**Lemma 3.1.** In equilibrium,

$$\tilde{\beta}_i = \left( \frac{p_i^y y_i}{P_c^y C} \right),$$

where $P_c$ is the ideal price index for the household, $C$ is total consumption by the household, $p_i$ is industry $i$’s ideal price index and $y_i$ is industry $i$’s composite output.

Note that in the Cobb-Douglas limit, where the elasticity of substitution across industries $\sigma$ is
equal to one, $\tilde{\beta}_i$ is precisely an industry’s share of sales. In the Cobb-Douglas case, $\tilde{\beta}$ coincides with
the influence measure defined by Acemoglu et al. (2012).

There are two reasons to think of $\tilde{\beta}$ as a supplier centrality. First, since it measures how
frequently an industry appears in other agents’ supply chains, it means that star suppliers have
high $\tilde{\beta}$. Secondly, as we shall see, $\tilde{\beta}$ captures the response of output to productivity shocks. For instance, Acemoglu et al. (2012) show that in a Cobb-Douglas model without an extensive
margin, $\tilde{\beta}$ captures the extent to which industry-specific labor productivity shocks affect output.
Furthermore, since $\tilde{\beta}$ is share of sales in a Cobb-Douglas economy, the work of Hulten (1978)
implies that $\tilde{\beta}$ maps marginal TFP shocks to aggregate output. The intuition for these results is clear: if a firm is supplying a large fraction of the economy, then its productivity shocks have a large impact on output.

Note two important facts about $\tilde{\beta}$. First, if there is no entry, so that $\tilde{M}$ is constant, then $\tilde{\beta}$ is exogenous with respect to productivity shocks. Second, note that $\tilde{\beta}_k$ depends *only* on industry $k$’s consumers, and not on industry $k'$’s suppliers. That is, two industries with the same demand-chain will have the same $\tilde{\beta}$ regardless of their own supply chains (I formally show this in the next section).

An analogous demand-side centrality measure can also be defined.

**Definition 3.2.** The consumer centrality is

$$\tilde{\alpha} = \Psi_d \alpha,$$

where

$$\Psi_d = (I - \mu^{1-\sigma} \tilde{M}^{\sigma-1} \Omega)^{-1} \mu^{1-\sigma} \tilde{M}^{\sigma-1} = \sum_{n=0}^{\infty} \left( \mu^{1-\sigma} \tilde{M}^{\sigma-1} \Omega \right)^n \mu^{1-\sigma} \tilde{M}^{\sigma-1}.$$

This is the flip-side to the supply-side centrality measure. It captures how frequently an industry appears in demand-chains. Whereas $\tilde{\beta}_k$ depended only on who bought from $k$, the consumer centrality $\tilde{\alpha}_k$ depends only on who $k$ buys from. Baqee (2014) shows that, in a model without an extensive margin, $\tilde{\alpha}$ captures the response of output to demand shocks. As the following lemma shows, consumer centrality $\tilde{\alpha}$ is a transformation of an industry’s price:

**Lemma 3.2.** In equilibrium,

$$\left( \frac{p_i}{w} \right)^{1-\sigma} = \tilde{\alpha}_i,$$

where $p_i$ is industry $i$’s price index and $w$ is the nominal wage.

Since prices are collinear with marginal costs, this means that $\tilde{\alpha}$ is a measure of marginal costs. This makes clear why $\tilde{\alpha}_k$ depends on industry $k$’s supply-chain, since the it is suppliers and not consumers, who contribute to marginal costs.

In defining the consumer centrality $\tilde{\alpha}$ and proving lemma 3.2, I have not made any reference to the productivity shocks $z$. As alluded to earlier, this is an abuse of notation, because I treat $\alpha_k$ as already incorporating the productivity shock. This is because when $\sigma \neq 1$, a productivity shock $z_k$ to industry $k$ is equivalent to changing that industry’s labor intensity from $\alpha_k$ to $\alpha_k z_k^{\sigma-1}$. Therefore, as long as $\sigma \neq 1$, we can think of $\alpha_k$ as including both the productivity shock and the labor intensity. When $\sigma = 1$, the consumer centrality is trivially equal to a vector of ones regardless of the productivity shocks.

A key result, that delivers much of the intuition of the results in the paper, is the following characterization of active firms’ profit functions in terms of supplier and consumer centrality measures:
Theorem 3.3. The payoffs of firm $i$ in industry $k$ are equal to

$$\pi(k, i) = \frac{1}{\varepsilon_k M_k} \times P^k \times \tilde{\beta}_k \times \tilde{\alpha}_k - w f_k.$$  

Note that $M_k \pi(k, i)$ gives industry $k$'s profits $\Pi_k$.

Without loss of generality, we can set the nominal wage $w = 1$. The expression in theorem 3.3 tells us that the profits of a firm are determined by a few intuitive key statistics. The product of $\tilde{\beta}_k$ and $\tilde{\alpha}_k$, which are the supply-side and demand-side centrality of the industry give us an industry's share of sales. The term $P^k C$ is an economy-wide shifter of all industry's profits, akin to aggregate demand. The division by $\varepsilon_k$ converts an industry's sales into profits since the within-industry elasticity of substitution determines mark-ups. Dividing gross industrial profits by the mass firms $M_k$ in that industry turns gross industrial profits into gross firm-level profits. Finally, we arrive at a firm’s profits by subtracting the fixed costs of entry from its gross profits.

Path Example

Before moving on to an analysis of the equilibrium, first let us demonstrate the intuition so far using a simple example of a production chain, shown in figure 1.

![Figure 1: A production path example. The solid arrows represent the flow of goods and services, and the dashed arrows indicate the flow of money. The household HH buys from industry 1 who buys from industry 2 and so on. Each industry in turn pays labor income and rebates profits to the household.](image)

Begin by computing the supplier-centrality for node $k$ in this chain:

$$\tilde{\beta}_k = \beta'(I - \tilde{M}^{-1})^{-1} \mu^{-\sigma} \Omega^{-1} e_k,$$

$$= \prod_{i=0}^{k-1} (1 - \alpha_i) M_i^{\frac{\beta_{i+1}}{\varepsilon_{i+1}}} \left(\frac{\varepsilon}{\varepsilon - 1}\right)^{-\sigma}.$$  

First, note that $\tilde{\beta}_k$ is a product. Therefore, if any industry $i < k$ disappears $M_i = 0$, then the supplier centrality of industry $k$ drops to zero. This intuitive, since if a downstream industry collapses, that
cuts all upstream industries off from any demand. The centrality of \( k \) as a supplier is increasing in the strength of its downstream connections \((1 - \alpha)\) and decreasing in the size of downstream markups \( \varepsilon_i / (\varepsilon_i - 1) \). The latter represents double-marginalization in this economy. Note that as long as the elasticity of substitution is greater than 1, an industry’s supplier centrality is increasing in the mass of downstream industries. Intuitively, when the elasticity of substitution is greater than one, more industries downstream attract more demand from the household. Lastly, observe that \( k \)th industry’s supplier centrality depends purely on its customers, customers’ customers, and so on. It does not depend on its suppliers. This is a general property of the supplier centrality, and we shall prove it in section 4.

Now, compute the consumer-centrality for node \( k \) in this chain:

\[
\tilde{\alpha}_k = e'_k (I - \bar{M}^{\sigma-1} \mu^{1-\sigma} \Omega)^{-1} \bar{M}^{\sigma-1} \mu^{1-\sigma} \alpha,
\]

\[
= \sum_{j=k+1}^N \left( \prod_{i=k}^{j-1} (1 - \alpha_i) M_i^{\alpha_i-1} \left( \frac{\varepsilon_i}{\varepsilon_i - 1} \right)^{1-\alpha} \right) \alpha_j + \alpha_k M_k^{\alpha_k-1} \left( \frac{\varepsilon_k}{\varepsilon_k - 1} \right)^{1-\sigma}.
\]

The consumer centrality is slightly more complex. It is an arithmetico-geometric. The intuition is that even if an upstream supplier \( i \) for industry \( k \) disappears, industry \( k \) still has access to all suppliers \( j \) where \( j < i \). But any supplier \( j > i \) also drops out. This gives rise to the arithmetic series, where each term is a product. Once again, the consumer centrality is increasing in the strength of the connection. And as long as the elasticity of substitution is greater than one, consumer centrality is decreasing in markups and increasing in the mass of upstream firms.

When the elasticity of substitution is less than one, consumer centrality is increasing in upstream markups. This sounds perverse until we recall that proposition 3.1 implies that

\[
\tilde{\alpha}_k = \left( \frac{p_k}{w} \right)^{1-\sigma}.
\]

Therefore, when the elasticity of substitution is less than one, higher consumer centralities indicate higher prices, not lower prices. Therefore, it is intuitive that in this case, higher upstream markups correspond to higher consumer centrality, since this corresponds to higher prices. Finally, note that, consumer centrality of a firm depends only on who it buys from, and not on who it sells to. Once again, this is a general property of consumer centrality that we discuss in more detail later.

**Analysis of the Full Equilibrium**

Our analysis of the equilibrium of this model, and its responses to the shocks, will proceed in parts. First, we fix the mass of firms in each industry to isolate the intensive margin responses. Firms may make nonzero profits in equilibrium, and the model is a generalization of the models in Acemoglu et al. (2012) or Long and Plosser (1983). With the entry margin shut down, I show
that the relevant notion of a firm’s systemic importance is exogenous and is approximated by its share of sales. Furthermore, a firm’s importance depends only on that firm’s role as a supplier of goods. A firm’s role as a consumer of inputs is irrelevant. Lastly, I show productivity shocks can never be amplified in this class of models. This is because a shock is always largest at its source – as it travels through the network, the shock is attenuated and the aggregate impact of the shock is a convergent geometric sum. The more influential the firm, the slower the decay. However, there are no cases where the shock actually gets bigger as it travels through the network.

After analyzing the model without entry, I allow free-entry but consider the limit of the model when firms have no mass, $\|\Delta\|\rightarrow 0$. In this case, systemic importance is endogenous and can change depending on economic conditions. Furthermore, importance is no longer well-approximated by size. Finally, a firm’s systemic importance depends not just on its importance as a supplier of inputs but also as a consumer of inputs. In this sense, both the in-degrees and out-degrees matter. However, in this limit, the model still cannot amplify shocks. As in the model without an extensive margin, shocks decay geometrically as they travel from the source.

Finally, I consider the case when $\|\Delta\| > 0$. This model features amplification mechanisms, and cascading failures, that are not present in the continuous model. This model can be thought of as an interpolation between discontinuous threshold models and continuous pulse models. To solve for the discontinuous model’s equilibrium, we need to solve a computationally intractable integer programming problem. Most of the time, this model’s behavior can be approximated by its continuous limit. However, in certain cases, its behavior is very different. This is because in some cases, a small shock can cause a systemically important firm to discontinuously exit. This can snowball into further failures and build on itself. I prove an inapproximability result that gives conditions for when the continuous model behaves very differently to the discrete model, along with some informative examples.

4 No Extensive Margin

In this section, we consider the behavior of this model when the extensive margin is shut down. This assumption means that this model is a generalization of the canonical input-output model of Long and Plosser (1983). I show that in this class of models, an industry’s influence on other industries and on the aggregate economy is exogenously determined, and that it is only dependent on how important the industry is as a supplier of inputs. Furthermore, I show that a firm’s size is an important determinant of a firm’s influence. Last, I show that this class of models is incapable of local amplification. A shock to a firm or industry will always decay as it travels from the source to its neighbors. This makes clear why systemically important industries must be large ones, since the decaying propagation of shocks means that a shock’s aggregate impact is necessarily limited by its immediate impact.

13
4.1 Perfect Competition

Standard models that lack an entry margin are typically perfectly competitive, with no fixed costs, so that the representative firm in each industry makes zero profits. To get to this benchmark, let \( f_i = 0 \) for every industry so that there are no fixed costs; and let \( \varepsilon_i \to \infty \) so that all industries are perfectly competitive and firms have no market power. Then, due to constant returns to scale, the size of firms in each industry is indeterminate. Without loss of generality, we can fix \( M = I \) so that there is a unit mass of firms in each industry. Technically, there may be entry or exit, but since firms in each industry are completely homogenous, entry and exit is observationally equivalent to firms getting larger or smaller. That is, the extensive and intensive margins are indistinguishable.

Note that in this special case \( \Psi_s = \Psi_d = (I - \Omega)^{-1} \). To simplify notation, let \( \Psi = (I - \Omega)^{-1} \). In subsection 4.2, I show how allowing for fixed costs and monopolistic competition, but not allowing entry, would affect the results of this section.

First, let us consider real GDP \( C \) as a function of productivity shocks \( z \) for a Cobb-Douglas economy \( E \).

**Proposition 4.1 (Productivity shock).** Let the elasticity of substitution across industries be equal to one, then

\[
\log(C(z|E)) = \tilde{\beta}'(\alpha \circ \log(z)) = \sum_{k=1}^{N} \frac{w_k}{GDP} \log(z_k).
\]

That is, the network-structure \( \Omega \) which has \( N^2 \) parameters is summarized by \( N \) sufficient statistics. These sufficient statistics are each industry’s expenditures on labor as a share of GDP.

This proposition is a slight generalization of results in Acemoglu et al. (2012). If we assume that each industry’s labor share is constant, so that \( \alpha_i = \alpha \) for all \( i \), and household expenditure shares are uniform so that \( \beta_i = 1/N \), then we exactly recover the result in Acemoglu et al. (2012), which tells us that the effect of a productivity shock on real GDP depends on share of sales.

**Corollary (Productivity shock).** Let the elasticity of substitution across industries be equal to one, and all industries have the same labor share \( \alpha \), then

\[
\log(C(z|E)) = \alpha \sum_{k=1}^{N} \frac{p_k y_k}{GDP} \log(z_k).
\]

That is, the network-structure \( \Omega \) which has \( N^2 \) parameters is summarized by \( N \) sufficient statistics. These sufficient statistics are each industry’s share of sales.

These propositions imply that when \( \sigma = 1 \), the supplier centrality \( \tilde{\beta} \) is a sufficient statistic for translating productivity shocks into real GDP. In particular, the aggregate impact of a vector of shocks depends on the supplier centrality weighted average of the productivity shocks. As
Lemma 3.1 shows, $\tilde{\beta}_k$ is equal to industry $k$’s share of sales. Therefore, the bigger the industry in equilibrium, the larger the impact of its productivity shocks on GDP. The exact nature of the network-structure is irrelevant since an industry $i$ could be big because it sells a lot to the household (large $\beta_i$) or because it supplies many other firms (the $i$th column of $\Psi$ is large).

These intuitions also carry over to the case where $\sigma \neq 1$. When the elasticity of substitution across industries $\sigma \neq 1$, productivity shocks to labor are isomorphic to changes in the labor share parameters $\alpha$. In particular, a productivity shock $z_k$ to industry $k$ is equivalent to changing industry $k$’s labor share parameter from $\alpha_k$ to $\alpha_k z^{\sigma-1}_k$. Therefore, we simply investigate changes to $\alpha$.

**Proposition 4.2** (Productivity shock). *When the elasticity of substitution across industries is not equal to one, then*

$$C(\alpha|E) = \left(\tilde{\beta}' \alpha\right)^{\frac{1}{\sigma-1}}.$$  

*That is, the network-structure $\Omega$, which has $N^2$ parameters, is summarized by $N$ sufficient statistics: $\tilde{\beta}$.  

Outside of the Cobb-Douglas case, the supply-side centrality $\tilde{\beta}' = \beta' \Psi_s$ is no longer an industry’s share of sales. Lemma 3.1 shows that it is still related to an industry’s share of sales however. In fact, even though $\tilde{\beta}'$ is not equal to share of sales globally, it is still closely related to share of sales.

**Proposition 4.3.** *Around the steady-state, where $z_k = 1$ for all $k$, we have that*

$$\frac{\tilde{\beta}_k}{\tilde{\beta}_l} = \frac{p_k y_k}{p_l y_l},$$

*so although $\tilde{\beta}$ is not equal to share of sales everywhere, at the steady-state, it does reflect an industry’s size.*

This is a consequence of that fact that in the steady-state with no markups or productivity shocks, all firms have the same price. And, as long as all firms have roughly the same price, we can interpret $\tilde{\beta}$ as being roughly equal to share of sales. Away from this steady state, $\tilde{\beta}$ is still the relevant sufficient statistic, although it is no longer equal to the share of sales.

Even though $\tilde{\beta}$ no longer corresponds to an industry’s size everywhere, it is still exogenous, and depends solely on the amount of household demand that reaches the industry, whether directly through retail sales, or indirectly through other industries. As before, the exact nature of the network-structure is irrelevant since an industry $i$ could be big because it sells a lot to the household (large $\beta_i$) or because it supplies many other firms (the $i$th column of $\Psi_s$ is large).

Furthermore, not only is this statistic exogenous with respect to productivity shocks, but it also depends only on the industry’s role as a supplier to other agents. If two industries sell the same amount to the same group of industries, they will have the same $\tilde{\beta}$ regardless of who they themselves are buying from. We can formally state this intuition as follows.
Proposition 4.4. Consider two industries $k$ and $l$ such that

$$\omega_{jk} = \omega_{jl}, \quad (j = 1, \ldots, N), \quad \text{and} \quad \beta_k = \beta_l,$$

then $\tilde{\beta}_k = \tilde{\beta}_l$. In other words, if two industries have the same immediate customer base, their supplier-centralities are the same.

In light of these propositions, we can state the following result.

Theorem 4.5. For every household share parameter vector $\beta$ and input-output share parameter matrix $\Omega$, there exists a different economy with household share parameter vector $\tilde{\beta}$ and degenerate input-output share parameter matrix $0$, such that

$$C(\alpha|\beta, \Omega) = C(\alpha|\tilde{\beta}, 0).$$

This result means that the existence of input-output connections is isomorphic to a change in household share parameters. This underlies the earlier claim that, it is not the interconnections themselves, but how intensively the household ultimately consumes goods from each industry that determines the model’s equilibrium responses to shocks.

Local Amplification

One reason why a firm’s size matters for its impact on aggregate outcomes is the continuous nature of shock propagation in this class of models. In input-output models, a shock to industry $k$ has the largest impact on industry $k$. The shock influences $k$’s neighbors, but the effect of the shock is attenuated by the weakness of its connections. The effect of the shock on $k$’s neighbors’ neighbors is further attenuated by the weakness of its connections’s connections. The shock decays geometrically with a decay rate controlled by the input-output matrix $\Omega$. This implies that the shock has its greatest impact at its source.

To show this, first we need the following technical lemma, which is a novel result in linear algebra.

Lemma 4.6. Let $A$ be a non-negative $N \times N$ matrix whose rows sum to one. Let $a, b, c$ be $N \times 1$ vectors in the unit cube with $c = a + b$. Let $B = (I - \text{diag}(1-c)A)^{-1}$. Then

$$\frac{e_i'Be_i}{e_i'Ba} \geq \frac{e_j'Be_j}{e_j'Ba} \quad (i \neq j),$$

where $e_i$ is the $i$th standard basis vector. When $a$ is strictly positive, then the inequality is strict.

The following proposition shows that the equilibrium effect of a productivity shock $z_i$ to industry $i$’s sales, in percentage change terms, is greatest for industry $i$. Furthermore, the effect on
other industries decays geometrically according to the sum of geometric matrix series \( \Psi \). Since 
sales are proportional to gross profits, the proposition also applies to gross profits.

**Proposition 4.7.** Consider the semi-elasticity of firm \( i \)'s sales relative to firm \( j \)'s sales with respect to a shock 
to firm \( i \)'s productivity around the steady state \( z = 1 \):

\[
\frac{d \log(sales_i)}{dz_i^{q^{-1}}} - \frac{d \log(sales_j)}{dz_i^{q^{-1}}} = \alpha_i \left( \Psi_{ii} - \Psi_{ji} \right) > 0.
\]

This proposition formalizes the intuition that a shock has to decay as it travels, and the input-output connections \( \Omega \) control the rate of decay. The rate of decay can be slow enough to overturn 
the law of large numbers, as shown by Acemoglu et al. (2012). However, even though the network 
can slow the decay, proposition 4.7 shows that shocks cannot be amplified as they travel out 
from their source. In order for shocks to industry \( k \) to have a big impact, industry \( k \) must have 
strong connections to the household, or strong connections to other industries that have strong 
connections to the household, and so on. But having strong connections to the household implies 
that the industry has to be large in equilibrium. The lack of local amplification means that size 
and influence are closely linked in this class of models. This, and the earlier results in this section, 
show that as long as firms are small, and they do not supply large fractions of the economy, then 
shocks to them cannot have significant aggregate effects.

### 4.2 No Extensive-Margin and Monopolistic Competition

So far, we have assumed that there are no fixed costs of entry and all industries are perfectly 
competitive. In this subsection, I consider the case where there is no entry but firms have some 
market power and positive fixed costs. This makes it easier to understand how adding the 
extensive margin will affect the results, since the extensive margin will only matter once we have 
fixed costs and product differentiation. I state a few key propositions in this subsection to show 
how the intuition of the previous section will carry over to this case.

First, let us consider the special case where the elasticity of substitution across industries \( \sigma = 1 \), 
which is the Cobb-Douglas case.

**Proposition 4.8 (Productivity shock).** Let the elasticity of substitution across industries be equal to one, 
them

\[
\log(C(z|E)) = \text{const} + \beta'(I - \Omega)^{-1}(\alpha \circ \log(z)).
\]

That is, the network-structure \( \Omega \), which has \( N^2 \) parameters, is summarized by \( N \) sufficient statistics. These 
sufficient statistics are \( \beta'(I - \Omega)^{-1} = \beta'\Psi_d \).

17
The sufficient statistic $\beta'\Psi_d$ is closely related to the sales shares $\tilde{\beta} = \beta'\Psi_s$. To see this, note that

$$\tilde{\beta} = \beta' \sum_{t=0}^{\infty} (\mu^{-1}\Omega)^t,$$

where $\mu$ is the diagonal matrix of mark-ups. Whereas,

$$\beta'\Psi_d = \beta' \sum_{t=0}^{\infty} (\Omega)^t.$$

Therefore, the relevant statistics $\beta'\Psi_s$ are the sales shares that would prevail if there were no mark-ups. Intuitively, they are still an exogenous supplier-centrality measure.

**Proposition 4.9 (Productivity shock).** When the elasticity of substitution across industries is not equal to one and there is no entry or exit, then

$$C(\alpha|E) = \frac{1 - Mf}{1 - \text{diag}(\varepsilon)^{-1}\tilde{\beta}'\tilde{\alpha}/\beta'\tilde{\alpha}} \cdot \frac{\tilde{\beta}'\tilde{\alpha}}{\beta'\tilde{\alpha}},$$

where $\varepsilon_k < \infty$, and endogenous expenditure shares $\sigma \neq 1$, each time funds flow from one industry to another, they are attenuated by monopoly profits. Longer chains accrue larger mark-ups, and this changes the expenditure shares. Due to this effect, we need a measure that not only takes the intensity of supply chains into account, but also their length. This is the role that the second sufficient statistic $\beta'\Psi_s\Psi_d = \tilde{\beta}'\Psi_d$ plays.

$\tilde{\beta}$ is just the supplier centrality we have dealt with previously. It is an exogenous supplier centrality, and depends solely on the amount of household demand that reaches the industry, whether directly through retail sales, or indirectly through other industries. As before, the exact nature of the network-structure is irrelevant since an industry could be big because it sells a lot to the household (large $\beta$) or because it supplies many other firms (large $\Psi_d\varepsilon_i$).

The Cobb-Douglas case constitutes a very special knife-edge scenario where, because expenditure shares are exogenous, the length of supply chains do not matter. Once we allow for market power $\varepsilon_k < \infty$, and endogenous expenditure shares $\sigma \neq 1$, each time funds flow from one industry to another, they are attenuated by monopoly profits. Longer chains accrue larger mark-ups, and this changes the expenditure shares. Due to this effect, we need a measure that not only takes the intensity of supply chains into account, but also their length. This is the role that the second sufficient statistic $\beta'\Psi_s\Psi_d = \tilde{\beta}'\Psi_d$ plays.

To see this, suppose that we have a unit mass of firms in each industry $M = I$ and that all mark-ups are zero. In this extreme case, $\Psi_d = \Psi_s$. So, we can interpret

$$\Psi_d\Psi_s = (I - \Omega)^{-2} = I + 2\Omega + 3\Omega^2 + 4\Omega^3 + \ldots.$$
controls for the length of the supplier relationship as well as its intensity.

Local Amplification

The following proposition shows that the equilibrium effect of a productivity shock $z_i$ to industry $i$’s sales, in percentage change terms, is greatest for industry $i$. Furthermore, the effect on other industries decays geometrically according to the sum of geometric matrix series $\Psi^s$. Since sales are proportional to gross profits, the proposition also applies to gross profits.

**Proposition 4.10.** For any economy $E$, if we fix the mass of firms in each industry, we have

$$\frac{d \log(sales_i)}{dz_i} - \frac{d \log(sales_j)}{dz_i} = \frac{\Psi^s_{ii} \alpha_i}{\bar{\alpha}_i} - \frac{\Psi^s_{ji} \alpha_i}{\bar{\alpha}_j} \geq 0.$$ 

5 Extensive Margin

Now that we understand the properties of the model without the extensive margin, let us consider how entry and exit changes the model’s properties. In this case, the network structure is endogenous since the number of firms in each industry is endogenous. This means that our notions of centrality $\tilde{\beta}$ and $\tilde{\alpha}$ are determined in equilibrium, and they change in response to shocks. It is no longer the case that an industry’s influence is exogenous, nor is it the case that only its role as a supplier matters. In other words, an industry’s influence depends both on its in-degrees and its out-degrees, and its degrees are determined in equilibrium depending on the mass of firms in other industries. Two firms with identical roles as suppliers can have markedly different effects on the equilibrium depending on their roles as consumers.

Our analysis will proceed in steps. First, we consider the limiting case of the model where all firms have zero mass. This means that the mass of firms in each industry will continuously adjust to ensure that all active firms make zero profits in equilibrium. This will allow for sharp analytical results about the equilibrium response to marginal shocks.

Once we have characterized the properties of the continuous limit, we consider the case where firms can have positive mass $\Delta > 0$. In this case, the model inherits some of the cascading and amplification properties of linear threshold models like Schelling (1971), but it also retains the linear geometric structure of Leontief (1936). Most of the time, the model can be expected to behave like its continuous limit; however, occasionally, this approximation can dramatically break down. The intuition is that each industry can support an integer number of firms. If firms are sufficiently small and there are many of them in an industry, then a shock to the industry will continuously perturb the mass of firms in that industry. However, if there are few firms in an industry, then shocks below a certain threshold are attenuated and will not result in any change to the number of active firms. However if shocks are bigger than that threshold, they will cause a
firm to discontinuously exit or enter. This adds an additional impulse to the original shock, which will now travel to the neighbors of the affected industry with more force. This process can feed on itself as a small firm’s failure can trigger a chain reaction that results in a large number of firms exiting.

A full characterization of the equilibrium of the model with lumpy firms is not possible. A natural solution is to use the continuous model to approximate the behavior of the model with lumpy firms. However, I show conditions under which the continuous limit provides a bad approximation to the discontinuous model. I show that the network can make firm’s payoffs non-monotonic in each other’s entry decisions. So, there are regions where entry decisions are strategic complements and and regions where they are substitutes. I show that when a shock puts us close to this intermediate region, the approximation error will explode.

To foreshadow the formal model, consider the extreme toy example in figure 2. The household HH is served by three industries: GM, F, and L. Two of these industries share a common supplier D. To make the intuition transparent, suppose that each industry consists of one firm. Recall that theorem 3.3 implies that in order for a firm to remain profitable, its sales $\tilde{\beta}_i, \bar{\beta}_i, \tilde{\alpha}_i$ have to be greater than an exogenous threshold $\epsilon_i$. Each term of $\tilde{\beta}_i, \bar{\beta}_i$, and $\tilde{\alpha}_i$ can respond to a shock. Therefore, the shock to a firm can travel via network connections $\tilde{\beta}$ and $\bar{\beta}$ or it can travel through household demand $P_i^c$.

The example in figure 2 demonstrates. In panel 2a, I show a negative shock to L’s fixed cost so that L is forced to exit. Then, through the effect of L on $P_i^c$, it can be the case that GM is forced to exit, and this is shown in panel 2b. GM exiting will cause D’s supplier centrality $\bar{\beta}_D$ to fall, and so D can be forced out, shown in panel 2c. Finally, once D is gone, this can cause F’s consumer centrality $\tilde{\alpha}$ to drop, which can cause them to also exit. Of course, this is not a realistic calibration of the model, and the assumptions of monopolistic competition are hard to justify with single-firm industries. However, this stark example does illustrate the forces operating in the model. In a canonical input-output model, like the one in section 4.2, a change to the fixed costs of L would have no network effects and simply lower real GDP $C$ by the amount of the fixed cost.

5.1 Massless Limit

To begin with, let us first consider the equilibrium of the model in the limit where $\Delta \rightarrow 0$. Computationally, this corresponds to the case where the mass of firms $M_k$ in industry $k$ will adjust so that firms in industry $k$ make exactly zero profits. This can be seen by taking the limit of the expression in definition 2.2 as $\Delta \rightarrow 0$. The primary motivation for looking at this limit is analytical tractability. By considering massless firms, we can glean useful intuition about the marginal effects of shocks on the equilibrium.

In this subsection, I show that once we allow for firm entry and exit, the model’s equilibrium responses can no longer be characterized in terms of sufficient statistics. The intuition is simple:
the centrality measures $\tilde{\beta}$ and $\tilde{\alpha}$ are functions of the masses of firms in each industry. Since the mass of firms responds to shocks, the centrality measures also respond to shocks. Furthermore, I show that an industry’s impact on supplier centralities depends on the industry’s own supplier centrality and its role as a consumer of inputs. On the other hand, an industry’s impact on consumer centralities depends on the industry’s own consumer centrality and its role as a supplier of inputs. So, although the supplier (consumer) centrality only depends on out-degrees (in-degrees), the way supplier (consumer) centrality changes in response to more entry depends on both the out-degrees and the in-degrees.

First to get intuition for the general model, let’s consider a very special case – the case where the elasticity of substitution is equal to one. In this case, all expenditure shares are exogenous. In this Cobb-Douglas case, we have the following result.

**Proposition 5.1.** When the elasticity of substitution is equal to one

$$C(z, f) = \tilde{\beta}' \left( \alpha \circ \log(z) - \frac{1}{\varepsilon - 1} \circ \log(f) \right) + \text{const.}$$

Therefore, the network-structure is summarized by N sufficient statistics $\tilde{\beta}$. Furthermore, in equilibrium

$$\tilde{\beta}_i = \frac{p_i y_i}{GDP}.$$ 

This shows that with Cobb-Douglas, the share of sales, which is exogenous, is again a sufficient statistic. Once again, the details and the complexity of the network are irrelevant, once we know
each industry’s share of sales. This is a knife-edge case. Once we deviate from Cobb-Douglas, expenditure shares respond to relative prices, and centrality measures become endogenous. This special case shows that the mechanism for our upcoming results depends crucially on the fact that expenditure shares respond to relative prices.

5.1.1 Out-of-Equilibrium Effect

To get intuition for the model’s properties, let’s consider the following out-of-equilibrium comparative statics: how does industry $k$’s supplier centrality change when there is entry in industry $i \neq k$? This comparative static holds fixed the mass of firms in all industries except industry $i$. In equilibrium, of course, the masses in other industries would respond, but it helps our understanding if we first isolate the partial equilibrium effect. If we had lags in entry and exit, these partial equilibrium results would be relevant for understanding short-run effects. Once we have characterized the out-of-equilibrium effects, we turn our attention to the equilibrium responses of the model.

**Lemma 5.2.** The derivative of $\tilde{\beta}_k$ with respect to a percentage change in the mass of firms in industry $i$, holding fixed the mass of firms in all other industries is given by

$$\frac{1}{M_i} \frac{\partial \tilde{\beta}_k}{\partial M_i} = \left( \frac{\sigma - 1}{\epsilon_i - 1} \right) \tilde{\beta}_i (\Psi_{ik} - 1(i = k)).$$

This expression is very intuitive. The impact to industry $k$’s centrality as a supplier depends on $i$’s importance as a supplier, and on how much $i$ buys from $k$ (whether directly or indirectly). So big effects are felt if $i$ is a key supplier and $i$ buys a lot from $k$. Note that if the elasticity of substitution $\sigma = 1$, then these derivatives are identically zero, which explains the neutrality result in proposition 5.1.

For ease of notation, let

$$\frac{\partial \tilde{\beta}}{\partial \log(Mi)} = \Psi_1,$$

then the result in lemma 5.2 can be written in matrix notation as

$$\Psi_1' = \text{diag}(\tilde{\beta}) \text{diag} \left( \frac{\sigma - 1}{\epsilon_i - 1} \right) (\Psi_s - I).$$

Similar results apply to the consumer centrality measure.

**Lemma 5.3.** The derivative of $\tilde{\alpha}_k$ with respect to a percentage change in the mass of firms in industry $i$, holding fixed the mass of firms in all other industries is given by

$$\frac{\partial \tilde{\alpha}_k}{\partial M_i} = M_i^{1-\sigma} \left( \frac{\sigma - 1}{\epsilon_i - 1} \right)^{\sigma-1} \tilde{\alpha}_k \Psi_{ki}^d,$$
This expression is the demand-side analogue to lemma 5.2. The impact to industry \( k \)'s centrality as a consumer depends on on \( i \)'s importance as a consumer, and on how much \( i \) sells to \( k \) (whether directly or indirectly). So big effects are felt if \( i \) is a key consumer and \( i \) sells a lot of \( k \). Once again, the impact to industry \( k \)'s consumer centrality depends on industry \( i \)'s suppliers and \( i \)'s customers.

For ease of notation, let
\[
\frac{\partial \tilde{\alpha}_k}{\partial \log(M_1)} = \Psi_2,
\]
then from lemma 5.3 we can write \( \Psi_2 \) in matrix notation,
\[
\Psi_2 = \Psi_d \text{diag}(\tilde{\alpha})\text{diag}\left(\frac{\sigma - 1}{\epsilon - 1}\right)\mu^{\sigma-1}\text{diag}(M)^{1-\sigma}.
\]

5.1.2 Path Example

Before moving on to an analysis of the equilibrium, first let us demonstrate the intuition of our partial equilibrium results using a simple example of a production chain depicted in figure 3.

Figure 3: The arrows represent the flow of goods and services. Let \( \omega_{k-1,k} \) be constant for all \( k \), and \( \beta_k, \alpha_k \) and \( \epsilon_k \) be constant for all \( k \).

In this example, each industry \( k \) sells some goods directly to the household, and some goods to the industry below it \( k - 1 \). For ease of exposition, consider \( \tilde{\beta} \) and \( \tilde{\alpha} \) at the point where all industries have a unit mass of firms. In this example, \( \tilde{\beta}_k \) is increasing in \( k \) and \( \tilde{\alpha}_k \) is decreasing in \( k \). That is, industry \( N \) is the most central supplier and least central consumer, and industry 1 is the most central consumer and least central supplier.

Now consider changing the mass of firms in industry \( N \). Industry \( N \) is the most central supplier so \( \tilde{\beta}_N > \tilde{\beta}_k \) for \( k \neq N \). However,
\[
\frac{\partial \tilde{\beta}_k}{\partial M_N} = 0,
\]
because industry \( N \) buys from no other industries. Therefore, its impact on supplier centralities is zero despite being the most central supplier.

Now consider changing the mass of firms in industry 1. Industry 1 is the most central consumer so \( \tilde{\alpha}_1 > \tilde{\alpha}_k \) for \( k \neq 1 \). However,
\[
\frac{\partial \tilde{\alpha}_k}{\partial M_1} = 0 \quad (k > 1),
\]
because industry 1 sells to no other industries. Therefore, its impact on consumer centralities is zero despite being the most central consumer.

This simple example illustrates why both the in-degrees and the out-degrees will matter for how the centrality measures will respond to a change in the mass of firms in each industry. Being a central supplier does not mean that entry or exit in your industry will have any effects on the supplier centralities of other industries. Similarly, being a central consumer does not imply that entry or exist in your industry will have any effects on the consumer centralities of other industries. To be influential, a firm must be central both as a consumer and as a supplier.

5.1.3 Equilibrium Impact of Shocks

So far, we have been focusing on out-of-equilibrium results. However, using lemmas 5.2 and 5.3, we can also analyse the general equilibrium impact of a productivity shock to an industry.

**Proposition 5.4.** The derivative of the equilibrium mass of firms in each industry $M$ with respect to a labor productivity shock in industry $k$ is given by

$$
\left(\frac{d \log(M)}{dz_k}\right) = (I - \text{diag}(\tilde{\beta})^{-1}\Psi_1 - \text{diag}(\tilde{\alpha})^{-1}\Psi_2)^{-1} \left(\frac{1}{P^C} \frac{dP^C}{dz_k} + \text{diag}(\tilde{\alpha})^{-1}\Psi d e_k\right).
$$

So we see that in equilibrium, a productivity shock to industry $k$ will first affect the masses in all other industries through its effect on the aggregate objects $P^C$ and through its effect on the marginal costs of anyone who buys from $k$. However, the initial change in masses results in the supplier and consumer centralities of all industries to change (captured by $\Psi_1$ and $\Psi_2$). This change in centrality measures, in turn, causes the masses to adjust again, and this changes the centralities again, and so on ad infinitum. This gives rise to a geometric series and the equilibrium effect is the sum of this geometric series.

For ease of notation, let

$$
\Lambda = \frac{\text{d} \log(\bar{\beta}) + \log(\bar{\alpha})}{\text{d} \log(M)} = \text{diag}(\bar{\beta})^{-1}\Psi_1 + \text{diag}(\bar{\alpha})^{-1}\Psi_2.
$$

Note that $\log(\bar{\beta}) + \log(\bar{\alpha})$ is proportional to each industry’s share of sales. Therefore, $\Lambda$ is the elasticity of industry sizes relative to the mass of firms in each industry. Now we can see the
intuition of 5.4 most clearly by expressing the derivative as

\[
\left( \frac{d \log(M)}{dz_k} \right) = \sum_{t=0}^{\infty} \Lambda^t \left( \frac{1 dP^e_i C}{P^e_i C} \right) + \text{diag} (\tilde{\alpha}^{-1} \Psi) e_k
\]

\[
= \sum_{t=0}^{\infty} \Lambda^t \left( \frac{1 dP^e_i C}{P^e_i C} \right) + \sum_{t=0}^{\infty} \Lambda^t \left( \text{diag} (\tilde{\alpha}^{-1} \Psi) e_k \right).
\]

The term

\( \text{diag} (\tilde{\alpha}^{-1} \Psi) e_k \)

is the intensive-margin effect of a productivity shock to industry \( k \). If industry \( k \) is more productive, that increases the productivity of any industry that buys inputs from \( k \). The degree to which an industry downstream from \( k \) is affected depends on how intensively it uses inputs from \( k \) (directly or indirectly). This is precisely the effect of a shock when the extensive margin is shut down as shown in proposition 4.10. Therefore, we can think of this as the traditional input-output effect.

With the extensive margin, the initial change in \( \tilde{\alpha} \) also causes the mass of firms to change. This change in the mass of firms causes further changes in the masses of firms. The cumulative effect on the equilibrium of these changes is captured by the “network-structure effect” term. The shock also has an effect on the general price level and real GDP, and the second “GE effect” captures this general equilibrium effect. It is a simple matter to show that in equilibrium, \( dP^e_i C/dz_k \) is just a weighted average of the network-structure effects.

As alluded to earlier, this complexity depends on the endogeneity of expenditure shares. For the Cobb-Douglas case, these formulas lose their interesting properties, and we only have the aggregate general equilibrium effects.

**Proposition 5.5.** If Cobb-Douglas, then the matrix is diagonal, and we have

\[
\frac{dM_i}{dz_k} = \frac{1}{\bar{\alpha}_i} + \frac{1}{\bar{\alpha}_k} + \frac{1}{M_i},
\]

\[
\frac{dM_j}{dz_k} = \frac{1}{\bar{\alpha}_j} + \frac{1}{\bar{\alpha}_k} + \frac{1}{M_j},
\]

because entry in industry \( i \) in response to shock to industry \( k \) is controlled just by exposure to aggregate objects and not via network interactions.

Now, let us analyze how real GDP responds to productivity shocks in this massless limit. This result should be compared to proposition 4.9, which gave the response when there was no extensive margin of entry and exit.
Proposition 5.6. With free entry,
\[
\left( \frac{d \log(C)}{dz_k} \right) = \text{scalar } \beta' \Psi_2 (I - \Lambda)^{-1} \left( \text{diag}(\bar{a})^{-1} + I \right) \Psi_d e_k \bar{\alpha}_k z_k^{\alpha - 2},
\]
where
\[
\text{scalar} = \left[ \beta' \bar{\alpha} + \beta' \Psi_2 (I - \Lambda)^{-1} 1 \right]^{-1}.
\]

It helps to compare this result to our earlier results. The presence of $\Lambda$, which depends on $\Psi_1$ and $\Psi_2$, and therefore depends on in-degrees and out-degrees, shows that the most influential industry is not simply the industry who supplies the most or consumes the most. Furthermore, all these matrices are now endogenous objects and depending on how many firms are in each industry, they will take different values. This means that the sufficient statistics approach of the earlier sections will no longer work. Furthermore, the influence measure

\[
\beta' \Psi_2 (I - \Lambda)^{-1} \left( \text{diag}(\bar{a})^{-1} + I \right) \Psi_d,
\]

which maps industrial shocks to movements in real GDP is not tied to sales.

To isolate the effect of the shocks to just the extensive margin, we could look at shocks to the fixed costs $f_k$ rather than to labor productivity. Such shocks only propagate through the network due to the change in masses, and therefore give rise to cleaner analytical expressions. However, to keep the results comparable to standard models, I have restricted my attention to productivity shocks (which have both intensive and extensive margin effects).

The results in this section show that the extensive margin is an important channel through which the network structure affects the equilibrium. However, despite the influence measures being endogenous and related in more realistic ways to the network-structure, this model is still incapable of generating local amplification.

Local Amplification

Now we can turn our attention to the behavior of a shock as it travels through the network. The equilibrium impact of a change in an industry’s productivity at the firm level is very stark.

Proposition 5.7. In equilibrium, the effect of a productivity shock $z_k$ on the sales of firm $i$ in industry $k$ is the same as its effect on firm $j$ in industry $l$.

\[
\frac{d \log(\text{sales}_{ki})}{dz_k} - \frac{d \log(\text{sales}_{lj})}{dz_k} = 0.
\]

At the firm level, the mass of firms in each industry adjusts to ensure that all firms are equally exposed to productivity shocks. At the industry-level, the conclusion is less stark.
Proposition 5.8. In equilibrium, the difference in the response of the profits of industry $k$ relative to industry $j$ to a productivity shock to industry $k$ is given by

$$\frac{d \log(s_k)}{dz_k} - \frac{d \log(s_j)}{dz_k} = (e_k - e_j)'(I - \Lambda)^{-1}\left(\text{diag}(\tilde{\alpha})^{-1}\Psi s e_k\right).$$

(1)

Numerical simulations suggest that, in equilibrium, this expression is always negative. In other words, the model is still incapable of amplifying shocks locally, and shocks decay as they move away from their source. The intuition for equation (1) is the following: The initial impact of a productivity shock to $k$, holding fixed the mass of firms in each industry, is given by $\text{diag}(\tilde{\alpha})^{-1}\Psi s e_k$ – this is the traditional input-output effect which captures the total intensity with which each industry uses inputs from industry $k$. However, with entry, the traditional input-output effect must be multiplied by a new term $(I - \Lambda)^{-1}$. This captures how the industry’s sizes change in response to the intensive margin shock. Recall that

$$\Lambda = \frac{\partial \log(\tilde{\beta}) + \log(\tilde{\alpha})}{\partial \log(M)} \propto \frac{\partial \log(\text{share of sales})}{\partial \log(M)},$$

where the proportionality sign follows from lemmas 3.1 and 3.2. Therefore, $(I - \Lambda)^{-1}$ captures the cumulative effect of the shock on the size of the industries.

5.2 Lumpy Firms

In this section, we consider the equilibrium where $\|\Delta\| > 0$. The equilibrium we focus on is the subgame perfect Nash equilibrium where firms make a simultaneous entry decision in period 1, and in period 2 they play monopolistic competition general equilibrium. Note that for tractability, I restrict the firms’ pricing strategies to take the aggregate industry price level and output to be exogenous.

An equilibrium will now feature as many firms in each industry as possible, in the sense that adding a firm to any industry would drive that industry’s profits to be negative.

Proposition 5.9. A mixed strategy equilibrium always exists.

Pure-strategy equilibria need not exist. Technically, the non-existence of a pure-strategy equilibrium means that this model of the economy has non-fundamental or non-exogenous randomness. Intuitively, pure-strategy equilibria can fail to exist due to cycling, where the entry of a firm causes the profits of another firm to go negative. Once that firm exists, another firm can enter that causes the profits of the first entrant to be negative, and so on.

Proposition 5.10. If $N > 2$, then a pure-strategy equilibrium does not always exist.

\(^1\)The proof of this result is work in progress.
We do not focus on non-existence or multiplicity of equilibria in this paper, although these issues are present. Instead, we focus on equilibria that, if they exist, converge to an equilibrium of the non-atomistic economy when we take the limit $\|\Delta\| \to 0$.

An equilibrium when $\|\Delta\| > 0$ is a solution to a nonlinear integer programming problem. Results from computational complexity theory show that we cannot hope to fully characterize the set of equilibria. A naive brute-force computation would become infeasible very rapidly, since with even a few hundred firms, we may need to consider more cases than there are atoms in the observable universe! Instead, our approach here will be to compare the discontinuous model’s equilibria to the equilibria of the continuous model. To demonstrate some of the subtleties, consider the following example illustrated in figure 4.

Figure 4: Two production lines selling to the household HH. Industries $2 - 6$ form on production line and industry 1 forms a second (degenerate) production line. The direction of arrows represent the flow of goods and services.

\[
\beta = (2/5, 2/5, 0, 3/5)', \\
f = (0.01, 0.01, 0.01, 0.01, 0.01, f_6), \\
\sigma = 1.1, \\
\varepsilon = (1.2, 3, 3, 3, 3, 3). \Delta = (0.5, 1, 1, 1, 1).
\]

Let us consider the equilibrium mass of entrants $M(\Delta, f_6)$ for different values of $f_6$ and mass vector $\Delta$. First, let us consider the massless limit,

\[
M(0, 0.04) = \begin{pmatrix} 15.7 \\ 17.6 \\ 10.1 \\ 5.8 \\ 3.3 \\ 1.2 \end{pmatrix}, \quad M(0, 0.05) = \begin{pmatrix} 16.9 \\ 16.2 \\ 9.4 \\ 5.3 \\ 3.0 \\ 0.8 \end{pmatrix}, \quad M(0, 0.06) = \begin{pmatrix} 17.8 \\ 15.4 \\ 8.7 \\ 4.9 \\ 2.7 \\ 0.6 \end{pmatrix}, \quad M(0, 0.07) = \begin{pmatrix} 18.7 \\ 14.6 \\ 8.2 \\ 4.6 \\ 2.5 \\ 0.5 \end{pmatrix},
\]
We see how increasing the fixed costs of the final supplier of the chain reduces the mass of firms active in the long chain and increases the mass of firms in the competing short chain in a continuous and intuitive way.

Now, let us consider this same equilibrium away from the massless limit.

$$M(\Delta, 0.04) = \begin{pmatrix} 16 \\ 17 \\ 10 \\ 5 \\ 3 \\ 1 \\ 0.5 \end{pmatrix}, \quad M(\Delta, 0.05) = \begin{pmatrix} 19 \\ 15 \\ 8 \\ 4 \\ 2 \\ 0.5 \end{pmatrix}, \quad M(\Delta, 0.06) = \begin{pmatrix} 19 \\ 14 \\ 8 \\ 4 \\ 2 \\ 0.5 \end{pmatrix}, \quad M(\Delta, 0.07) = \begin{pmatrix} 33 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

In going from $f_6 = 0.04$ to $f_6 = 0.05$, the discontinuous model amplifies the impact of the shock because the shock is large enough to force a discontinuous change. From $f_6 = 0.05$ to $f_6 = 0.06$ however, the discontinuous model attenuates the impact of the shock since the firms are large enough that they can absorb the losses without exiting. This represents a phase transition since shocks below a threshold are attenuated, and above that threshold, they are amplified. In going from $f_6 = 0.06$ to $f_6 = 0.07$, the interior equilibrium disappears, and the long supply chain collapses. Furthermore, this is the unique equilibrium, so this is not a coordination failure resulting from equilibrium switching. It is worth noting that in this example, if firms in industry 2 could commit to restrict entry, their supply chain would survive.

This example illustrates that most of the time, we can expect the model to behave like its continuous limit. However, some shocks are amplified when a discontinuous change occurs and other shocks are attenuated. Furthermore, occasionally, when whole industries exit, the discontinuous equilibrium can be very different. These differences are not only large, but they can also be very complex.

### 5.3 Approximation Error

To formalize the approximation error, we need a few definitions. These definitions have some resemblance to the concepts of natural friends and natural enemies in international trade theory. Motivated by the Stolper and Samuelson (1941) result, Deardorff (2006) defines an industry to be a natural friend (enemy) of a factor when an increase in its prices will increase (decrease) the returns to that factor. Inspired by these terms, I define the following.

**Definition 5.1.** Let $\pi_i(M_1, \ldots, M_N)$ be the profit function of industry $i$. Industry $j$ is an enemy of industry $i$ when

$$\frac{\partial \pi_i}{\partial M_j} < 0,$$
and industry \( j \) is a friend of industry \( i \) when
\[
\frac{\partial \pi_i}{\partial M_j} > 0.
\]
Industry \( j \) is a frenemy of industry \( i \) when
\[
\frac{\partial \pi_i}{\partial M_j}
\]
takes both positive and negative values.

Although the definition is reminiscent of the one in Deardorff (2006), these definitions are *out-of-equilibrium*. In other words, we change the mass of firms in one industry without changing the masses in other industries.

When industries are frenemies, the entry decisions of firms in one industry are strategic substitutes for some regions and strategic complements for other regions. I show that frenemies can only occur with non-degenerate (non-diagonal) input-output connections.

**Proposition 5.11.** Let \( \sigma > 1 \). When the network structure \( \Omega \) is degenerate (diagonal), all industries are enemies. When the network is non-degenerate, industries can be frenemies.

We can say more than this.

**Proposition 5.12.** If industry \( j \) is a frenemy of industry \( i \), then initially industry \( i \)'s profits are increasing in \( M_j \) and eventually industry \( i \)'s profits are decreasing in \( M_j \).

It turns out that we can guarantee that approximating the discontinuous model with a continuous limit will be bad when firms are frenemies.

**Theorem 5.13.** Let \( M(\Delta) \) correspond to the mass of firms in each industry in an equilibrium where firm-level masses are given by \( \Delta \). Let \( M(0) \) denote equilibrium masses in each industry when \( \Delta \to 0 \). Then
\[
\| M(\Delta) - M(0) \| \geq \| D\pi(\tilde{M}) \|^{-1}\| \pi(M(\Delta)) \|,
\]
where \( \tilde{M} \in \text{co}\{M(0), M(\Delta)\} \). Here, \( \text{co} \) refers to the convex hull and \( D\pi \) to the derivative of \( \pi(M) \) as a function of \( M \).

In particular, the error gets larger when the profit functions have flat slopes, which can only occur if industries are frenemies. When \( \sigma > 1 \), this can only happen with network connections, because a model with no network has monotone profit functions. The network makes the profit functions non-monotonic even when industry outputs are substitutes \( \sigma > 1 \).
6 Bail-outs and the Nature of Externalities

The examples in the previous section suggest that bail-outs and government interventions may be desirable, since the failure of an important firm or industry can take down entire parts of a network. However, figuring out which failures are efficient and which ones are not is impracticable. To implement the optimum, a social planner would not only need access to an infeasible amount of information, but it would also have to solve an intractable computational problem.

In this section, we imagine a scenario where the policy-maker can, in response to a shock, ask firms if other firms should be rescued. The industries will truthfully tell the policy-maker whether a rescue would increase their profits or decrease their profits. Under this assumption, we can investigate conditions under which the profits of an industry align with those of society.

This exercise is inspired by the example in the introduction, where the president of Ford Motor Company testified in favor of General Motors. The results in this section give sufficient conditions under which the policy-maker can trust a firm’s recommendation. The results of this section should not be taken literally as a guide to policy. Rather, this thought experiment tells us about the nature of externalities in this model. The aim of this section will be to prove a series of “innocent by-stander” results, which imply that firms’ requests for the bail-outs of other firms can only be trusted if the firms are not reliant on one another.

To start with, we need to define the following notion of connectedness.

**Definition 6.1.** Two firms $u$ and $v$ are connected if there exists a directed path from $u$ to $v$ or a directed path from $v$ to $u$, on a directed graph defined by $\Omega$.

This lemma will be the workhorse result for the rest of the section.

**Lemma 6.1.** If firm $u$ and firm $v$ are not connected, then in the event that $u$ fails, and we hold fixed the number of firms in all other industries, $\tilde{\beta}_v$ and $\tilde{\alpha}_v$ remain constant.

Now we are in a position to state our first result.

**Theorem 6.2.** Let $B$ be the set of firms not connected to $v$. If all firms in $B$ prefer for $v$ to be rescued, then it is Pareto-efficient for $v$ to be rescued. If the firms in $B$ disagree with each other, it must be because the firms who want $v$ to be rescued are badly affected through a cascade.

Consider the example in figure 5. Theorem 6.2 implies that in the event that GM, and only GM, is about to fail, Ford, Toyota, and Mitsubishi will agree with each other about whether or not GM should be rescued. Furthermore, if they agree that GM should be rescued, then the rescue is Pareto-efficient. However, if Ford, Toyota, and Mitsubishi disagree, then it must be that Ford is adversely affected by GM’s failure through the failure of their common supplier Dunlop Tires. Furthermore, the efficiency of the bailout is ambiguous. The welfare implication of theorem 6.2 is that Ford’s plea that GM be bailed out can only be trusted if Ford is worried about aggregate...
Figure 5: A stylized example of a supply chain. HH is household; GM is General Motors; F is Ford Motor Company; D is Dunlop Tires; M is Mitsubishi Motors; T is Toyota; and, B is Bridgestone Tires, a Japanese tire manufacturer. The arrows represent the flow of goods and services.

demand, not cascades. In his December 2008 testimony to Congress, Mulally referred to both the general economic downturn, as well as their overlapping supply base, as the reason why GM and Chrysler should be bailed out.

We can sharpen theorem 6.2 into the following “innocent by-stander” principle.

**Theorem 6.3.** Let \( u \) and \( v \) be two firms. Suppose that the only undirected path from \( u \) to \( v \) goes through the household. Then if \( u \) prefers for \( v \) to be rescued, rescuing \( v \) increases the utility of the household.

Note that theorem 6.3 considers *undirected* paths, meaning that we disregard the direction of the arrows. So, while \( F \) and \( GM \) are unconnected according to definition 6.1, there does exist an undirected path from \( F \) to \( GM \) that does not go through the household. Theorem 6.3 implies that, in the example of figure 5, Ford’s profits may not be aligned with societal preferences. However, the profits of Ford may be aligned with society when considering the bail-out of an unconnected firm like Lehman Brothers. That is, in a network like the one in figure 6, if a shock to L causes F’s profits to go down, we can safely infer that we should have bailed L out.

Unfortunately, the cases where we can trust a firm are precisely the cases where we would expect that firm to not be well-informed about the consequences of a failure. These results cast doubt on the idea that we reliably implement bailout policy by surveying a firm’s direct rivals.

**Bounding Inefficiency**

Although the asking mechanism I investigate is simple to understand, it does not necessarily achieve first-best. In particular, asking is sufficient but not necessary for an efficient bail-out. We can bound the losses from using this simplistic mechanism relative to first best, or other, more elaborate mechanisms that can implement first-best.
Theorem 6.4. The fraction of utility lost by asking mechanism relative to utility maximizing

\[
\text{fraction of utility lost} \leq \left( \frac{w_{fb} / P_{fc}^{fb}}{w_c / P_c^A} \right)^\sigma - 1,
\]

where \( P_c^A \) is the consumer price index under the asking equilibrium and \( P_c^{fb} \) is the consumer price index under the first best equilibrium.

This theorem give us a sense of how large the “mistakes” will be in terms of the real wage.

7 Conclusion

This paper highlights the importance of firm entry and exit in propagating and amplifying shocks in a production network. I show that without the extensive margin, canonical input-output models have several crucial limitations. First, a firm’s influence depends solely on the firm’s role as a supplier to other firms, and its role as a consumer is irrelevant. Second, each firm’s influence measure is exogenous, and the exogenous influence measures are sufficient statistics for the input-output matrix. In this sense, for every input-output model, there exists a non-interconnected model with different parameters that has the same equilibrium responses. Third, a firm’s influence is well-approximated by the firm’s size, so the nature of interconnections does not matter as long as it results in the same distribution of firm sizes. Finally, I show that canonical input-output models lack the ability to locally amplify shocks. That is, a shock to one industry always has the largest impact at its source, and its effect decays geometrically as it travels away from its source. This
shows why a firm has to be large in order to have a meaningful aggregate impact.

However, when the mass of firms in each industry can adjust endogenously, all of these properties disappear. The influence of an industry is endogenous, depends on its role as a supplier and as a consumer of inputs, and is not well approximated by its size. Two industries with the same demand-chains can have very different effects on the equilibrium if their supply-chains are different, and vice versa. Furthermore, there are no sufficient statistics that summarize the network. In this sense, the model is not isomorphic to a non-network model with different parameters.

Despite these features, I show that in the limit where all firms have zero mass, the model with entry still lacks the ability to locally amplify shocks. The size of each industry responds smoothly to shocks and the effects still decay geometrically as a shock travels from its source. However, when firms in some industries are “granular,” then a failure of one of these firms can snowball into an avalanche of failures. I show that we can expect these cascades to occur when firms’ entry decisions switch from being strategic substitutes to being strategic complements.

Finally, motivated by the possibility of catastrophic failure in this model, I show conditions under which the objectives of a firm are aligned with those of society. In principle, this would allow a policy-maker to formulate bail-out policy by surveying other firms. Unfortunately, the results show that we should not be very optimistic about this strategy, since the only circumstances when firms are trustworthy are precisely those circumstances where we would expect them to be as uninformed as the policy-maker. This is because a firm’s incentives are only aligned with society when a firm’s exposure to the failure of the other firm is through general equilibrium effects. In other words, in cases where the firm is not directly linked to the troubled industries. Despite this, these results do show that the welfare consequences of shocks that travel from one firm to another are different depending on whether they arrive via network connections or general equilibrium effects.
References


8 Appendix I: Proofs

Lemma 8.1. Demand for firm $i$ in industry $k$’s output is

\[ y(k, i) = c(k, i) + \sum_l \int_{M_l} x(l, j, k, i) d j, \]

\[ = \beta_k \left( \frac{p(k, i)}{p_k} \right)^{-\epsilon_k} \left( \frac{p_k}{p_c} \right)^{-\sigma} C + \sum_l M_l \omega_l \left( \frac{p(k, i)}{p_k} \right)^{-\epsilon_k} \left( \frac{p_k}{\lambda_l} \right)^{-\sigma} y(l, j). \]

Proof. Cost minimization by each firm implies firm $j$ in industry $l$’s demand for inputs from firm $i$ in industry $k$ is given by

\[ x(l, j, k, i) = \omega_l \left( \frac{p(k, i)}{p_k} \right)^{-\epsilon_k} \left( \frac{p_k}{\lambda_l} \right)^{-\sigma} y(l, j), \]

where $\lambda_l$ is the marginal cost of firms in industry $l$,

\[ \lambda_l = \left( \alpha_k \omega_l^{-1} w^{-1-\sigma} + \sum_l \omega_l p_l^{-1-\sigma} \right)^{\frac{1}{1-\sigma}}, \]

and $p_k$ is the price index for industry $k$

\[ p_k = \left( \int_{M_k} p(k, i)^{1-\epsilon_k} d i \right)^{\frac{1}{1-\epsilon_k}}. \]

Household demand for goods from firm $i$ in industry $k$ are

\[ c(k, i) = \beta_k \left( \frac{p(k, i)}{p_k} \right)^{-\epsilon_k} \left( \frac{p_k}{P_c} \right)^{-\sigma} C, \]

where $P_c$ is the consumer price index

\[ P_c = \left( \sum_k \beta_k p_k^{1-\sigma} \right)^{\frac{1}{1-\sigma}}. \]

Adding the household and firm’s demands together gives the result.

Proof of Lemma 3.1. By Lemma 8.1

\[ y(k, i) = \beta_k p(k, i)^{\epsilon_k} p_k^{\epsilon_k-\sigma} p_c^{-\sigma} C + \sum_l M_l p(k, i)^{\epsilon_k} p_k^{\epsilon_k-\sigma} \lambda_l^{\sigma} \omega_l y(l, j). \]
Substitute $p_k = M_k^{1/\varepsilon} p(k, i)$ to get

$$M_k^{1/\varepsilon} p_k^\varepsilon y(k, i) = \beta_k p_c^\varepsilon C + \sum_l M_l \lambda_l^\varepsilon \omega_{lk} y(l, j).$$

Observe that

$$y_k = \left( \int_{M_k}^{1/\varepsilon} y(k, i) \frac{1}{\varepsilon} \, di \right)^{\varepsilon - 1} = M_k^{1/\varepsilon} y(k, i).$$

Substitute this into the previous equation to get

$$p^\varepsilon y_k = \beta_k p_c^\varepsilon C + \sum_l \omega_{lk} M_l^{1/\varepsilon} \left( \frac{\varepsilon_l}{\varepsilon_l - 1} \right)^{-\varepsilon} p_l^\varepsilon y_l.$$

Define $\tilde{M}$ to be the diagonal matrix whose $k$th diagonal element is $M_k^{1/\varepsilon}$, and $\mu$ to be the diagonal matrix whose $k$th element is industry $k$'s markup $\varepsilon_k / (\varepsilon_k - 1)$. Now denote $s_k = p_k^\varepsilon y_k$. This means that we can write

$$s' = \beta' P_c^\varepsilon C + s' \tilde{M}^{-1} \mu^{-\varepsilon} \Omega.$$

Rewrite this to get

$$s' = \beta' (I - \tilde{M}^{-1} \mu^{-\varepsilon} \Omega)^{-1} P_c^\varepsilon C$$

$$= \tilde{\beta}' P_c^\varepsilon C.$$

\[\blacksquare\]

\textbf{Proof of lemma 3.2.} Since all firms have constant returns to scale on the margin, firm $i$'s problem in industry $k$, conditional on entry, can be written as

$$\max p(k, i) y(k, i) - \lambda_k y(k, i),$$

where, by lemma 8.1,

$$y(k, i) = \text{const} p(k, i)^{-\varepsilon_k},$$

because the firm does not internalize its effects on the aggregate price indices. This optimization problem gives

$$p(k, i) = \frac{\varepsilon_k}{\varepsilon_k - 1} \lambda_k.$$

So, as in standard monopolistic competition models, markups are constant.

Note that

$$p_k = \left( \frac{\varepsilon_k}{\varepsilon_k - 1} \right) M_k^{1/\varepsilon} \lambda_k.$$
Substituting this into the definition of $\lambda_k$ we get

$$\left(\frac{\varepsilon_k - 1}{\varepsilon_k}\right) M_k^{\frac{1}{\varepsilon_k}} p_k = \left(\alpha_k w^{1-\sigma} + \sum_l \omega_k \beta_l^{1-\sigma}\right)^{1/\varepsilon_k}.$$  

A column vector an exponent denotes element-wise exponentiation. Then if we let $P$ be the vector of $p_k^{1-\sigma}$ and $\alpha$ the vector of $\alpha_k$, then this system of equations can be written as

$$\mu^{\sigma-1} \tilde{M}^{1-\sigma} P = \alpha w^{1-\sigma} + \Omega P.$$  

We can rearrange this to get

$$P = \mu^{1-\sigma} \tilde{M}^{\sigma-1} \alpha w^{1-\sigma} + \mu^{1-\sigma} \tilde{M}^{\sigma-1} \Omega P.$$  

Rearrange this to get

$$P = (I - \mu^{1-\sigma} \tilde{M}^{\sigma-1} \Omega)^{-1} \mu^{1-\sigma} \tilde{M}^{\sigma-1} \alpha w^{1-\sigma} = \tilde{\alpha} w^{1-\sigma}.$$  

This implies that for each industry $k$

$$\left(\frac{p_k}{w_k}\right)^{1-\sigma} = \tilde{\alpha}_k.$$  

**Proof of theorem 3.3.** Note that the profits of firm $i$ in industry $k$ are

$$\pi(k, i) = p(k, i) y(k, i) - \lambda_k y(k, i) - w f_k.$$  

This is equivalent to

$$\pi(k, i) = p(k, i) y(k, i) - \frac{\varepsilon_k - 1}{\varepsilon_k} p(k, i) y(k, i) - w f_k,$$

$$= \frac{1}{\varepsilon_k} p(k, i) y(k, i) - w f_k.$$  

Since all active firms in industry $k$ are identical this is

$$\pi(k, i) = \frac{1}{\varepsilon_k M_k} p_k y_k - w f_k.$$  

By lemmas 3.1 and 3.2,

$$p_k y_k = p_k^{\alpha} y_k p^{1-\sigma} = \tilde{\beta}_k \tilde{\alpha}_k P^* C w^{1-\sigma},$$
and so
\[ \pi(k, i) = \frac{1}{\varepsilon_{kM_k}} \beta_k \tilde{\alpha}_k P_c^\sigma C W^{1-\sigma} - w f_k. \]

**Proof of proposition 4.1.** Note that real GDP can be written as
\[ C = \frac{P_c C}{P_c} = \frac{w l + \pi}{P_c}, \]
where \( \pi \) is total profits. By free entry, profits are zero in equilibrium. Normalize \( w = 1 \). Then
\[ \log(C) = -\log(P_c). \]

The marginal costs of firms in industry \( k \) are given by
\[ \lambda_k = \left( \frac{z_k}{w} \right)^{-\alpha_k} \prod_l p_l^{\omega_l}. \]

Since industries are perfectly competitive, firms set prices equal to their marginal costs. Let \( P \) denote the vector of industry prices. Then, in equilibrium,
\[ \log(P) = (I - \Omega)^{-1} (\alpha \circ \log(w) - \log(z)). \]

Therefore,
\[ \log(C) = -\log(P_c), \]
\[ = -\beta' \log(P), \]
\[ = -\beta'(I - \Omega)^{-1} (\alpha \circ \log(z)) \]
\[ = \tilde{\beta}' \alpha \circ \log(z). \]

Note that \( \tilde{\beta} \) is sales as a share of GDP, and \( \tilde{\beta}_k \alpha_k \) is therefore industry \( k \)’s wage bill as a share of GDP. \( \blacksquare \)

**Proof of proposition 4.3.** Note that industry \( k \)’s sales are given by
\[ p_k y_k = p_k^\sigma y_k = \tilde{\beta}_k \tilde{\alpha}_k P_c^\sigma C W^{1-\sigma}. \]

Observe that in equilibrium with no shocks,
\[ \tilde{\alpha} = (I - \Omega)^{-1} \alpha = 1. \]
Therefore, 
\[ p_k y_k = \tilde{\beta}_k P^\alpha C w^{1-\alpha}. \]
So an industry’s shares of sales relative to other industries is determined solely by \( \tilde{\beta}_k \).

**Proof of theorem 4.5.** Real consumption is given by 
\[ \frac{wl + \pi}{P_c} = \frac{1}{P_c}. \]

We have shown that 
\[ P_c = (\beta'(I - \Omega)^{-1}) \tilde{\beta}. \]
Let \( \tilde{\beta}' = \beta'(I - \Omega)^{-1} \) to get the desired result.

**Proof of proposition 4.7.** The sales of industry \( j \) are given by 
\[ \tilde{\beta}_j e_j' \Psi \left( \alpha \circ z^{\sigma-1} \right) P^\alpha_c. \]

Therefore 
\[ \frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} = \frac{1}{\tilde{\beta}_j} \frac{d \tilde{\beta}_j}{dz_i^{\sigma-1}} + \frac{1}{\tilde{\alpha}_j} \frac{d e_j' \Psi \left( \alpha \circ z^{\sigma-1} \right)}{dz_i^{\sigma-1}} + \frac{1}{P^\sigma_c} \frac{d P^\sigma_c C}{dz_i^{\sigma-1}} \]
\[ = 0 + \frac{1}{\tilde{\alpha}_j} \frac{d e_j' \Psi \left( \alpha \circ z^{\sigma-1} \right)}{dz_i^{\sigma-1}} + \frac{1}{P^\sigma_c} \frac{d P^\sigma_c C}{dz_i^{\sigma-1}}. \]

Note that 
\[ \frac{d \Psi \left( \alpha \circ z^{\sigma-1} \right)}{dz_i^{\sigma-1}} \bigg|_{z=1} = \Psi \frac{d \left( \alpha \circ z^{\sigma-1} \right)}{dz_i^{\sigma-1}} \bigg|_{z=1} = \Psi \alpha_i e_i. \]

Substitute this in to get 
\[ \frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} = e_j' \Psi \alpha_i e_i + \frac{1}{P^\sigma_c} \frac{d P^\sigma_c C}{dz_i^{\sigma-1}} \]
where we use the fact that \( \tilde{\alpha} = \tilde{\beta} \). This means that 
\[ \frac{d \log(\text{sales}_i)}{dz_i^{\sigma-1}} - \frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} = \alpha_i \left( e_i' \Psi e_i - e_j' \Psi e_j \right), \]
as required. Lemma 4.6 then implies that this is always greater than zero.
Proof of proposition 4.8. By theorem 3.3,
\[ \sum_k \pi_k = \frac{\tilde{\beta}'}{\varepsilon} \tilde{\alpha} P_c C - 1'M f, \]
where division by $\varepsilon$ is elementwise. Observe that
\[ P_c C = (1 + \sum_k \pi_k). \]
Therefore,
\[ \sum_k \pi_k = \left( \frac{\tilde{\beta}'}{\varepsilon} \tilde{\alpha} - 1'M f \right) \frac{1}{1 - \frac{\tilde{\beta}'}{\varepsilon} \tilde{\alpha}}. \]
Therefore, nominal GDP
\[ P_c C = 1 + \left( \frac{\tilde{\beta}'}{\varepsilon} \tilde{\alpha} - 1'M f \right) \frac{1}{1 - \frac{\tilde{\beta}'}{\varepsilon} \tilde{\alpha}}, \]
does not respond to shocks. Therefore, real GDP is given by
\[ \log(C) = \text{const} - \log(P_c). \]
Since
\[ \log(P_c) = -\beta'(I - \Omega)^{-1}(\alpha \circ \log(z)) + \text{const}, \]
we can write
\[ \log(C) = \text{const} + \beta'(I - \Omega)^{-1}(\alpha \circ \log(z)). \]
Proof of proposition 4.9.
\[ \sum_k \pi_k = \frac{\tilde{\beta}'}{\varepsilon} \tilde{\alpha} P^o_c C - 1'M f, \]
where division by $\varepsilon$ is elementwise. Observe that
\[ P^o_c C = P_c C P^o - 1 = \frac{(1 + \sum_k \pi_k)}{\tilde{\beta}' \tilde{\alpha}}. \]
Combine these two expressions to get the desired result.

Proof of proposition 4.10. Using the same argument as in the proof of proposition 4.7, we can show
that Substitute this in to get
\[
\frac{d \log(\text{sales}_j)}{dz_{j}^{\sigma - 1}} = \frac{1}{\bar{\alpha}_j} e_j' \Psi \alpha_i e_i + \frac{1}{P_c^C C} \frac{dP_c^C}{dz_{i}^{\sigma - 1}},
\]
This means that
\[
\frac{d \log(\text{sales}_i)}{dz_{i}^{\sigma - 1}} - \frac{d \log(\text{sales}_j)}{dz_{j}^{\sigma - 1}} = \alpha_i \left( \frac{e_i' \Psi e_i}{\bar{\alpha}_j} - \frac{e_j' \Psi e_j}{\bar{\alpha}_j} \right),
\]
as required. Lemma 4.6 then implies that this is always greater than zero. 

Proof of proposition 5.1. Note that real GDP can be written as
\[
C = \frac{P_c C}{P_c} = \frac{w l + \pi}{P_c},
\]
where \( \pi \) is total profits. By free entry, profits are zero in equilibrium. Normalize \( w = 1 \). Then
\[
\log(C) = - \log(P_c).
\]
The marginal costs of firms in industry \( k \) are given by
\[
\lambda_k = \left( \frac{\alpha_k z_k}{w} \right)^{-\alpha_k} \prod_{l} \left( \frac{\alpha_{kl} p_l}{p_l} \right)^{-\alpha_{kl}},
\]
substitute
\[
\lambda_k = M_k^{\frac{1}{\epsilon_k}} \frac{1}{\epsilon_k} - \frac{1}{\epsilon_k} p_k,
\]
and let \( P \) denote the vector of industry prices. Then, in equilibrium,
\[
\log(P) = (I - \Omega)^{-1} \left( -\alpha \circ \log(z) + \log(\mu 1) - \log(\tilde{M} 1) \right).
\]
Free entry implies that
\[
\tilde{M}_k = \left( \frac{\tilde{p}_k P_c C}{f_k \epsilon_k} \right)^{\frac{1}{\epsilon_k} - 1}.
\]
Substituting this in to \( \tilde{M} \) and combining with the fact that
\[
\log(P_c) = \beta' \log(P),
\]
gives
\[
\log(C) = - \log(P_c) = \beta' \alpha \circ \log(z) - \sum_k \frac{1}{\epsilon_k - 1} \tilde{p}_k \log(f_k) + \text{const},
\]
where
\[
\text{const} = \hat{\beta}' \left( \log(\mu_1) + \frac{1}{\epsilon - 1} \circ \log(\hat{\beta}) - \log(\epsilon) \right).
\]

In the above expression \(1/(\epsilon - 1)\) is the vector of \(1/(\epsilon_k - 1)\). By lemma 3.1, \(\hat{\beta}\) is proportional to the sales vector and nominal GDP is always equal to 1, we have the desired result. ■

**Proof of lemma 5.2.** Recall that
\[
\tilde{\beta}' = \beta'(I - \tilde{M}^{\sigma - 1} \mu^{1 - \sigma} \Omega)^{-1}.
\]

So
\[
\frac{d\tilde{\beta}'}{d \log(M_i)} = M_i \frac{d\beta'}{dM_i} = -M_i \beta' \Psi_s \frac{d(I - \tilde{M}^{\sigma - 1} \mu^{1 - \sigma} \Omega)}{dM_i} \Psi_s,
\]
\[
= M_i \beta' \Psi_s \frac{d\tilde{M}^{\sigma - 1} \mu^{1 - \sigma} \Omega}{dM_i} \Psi_s,
\]
\[
= M_i \beta' \Psi_s \frac{d\tilde{M}^{\sigma - 1} \tilde{M}^{1 - \sigma} \mu^{1 - \sigma} \Omega}{dM_i} \Psi_s,
\]
\[
= M_i \beta' \Psi_s \frac{d\tilde{M}^{\sigma - 1}}{dM_i} (\Psi_s - I),
\]

The \(k\)th element of this vector is
\[
\frac{d\tilde{\beta}_k}{d \log(M_i)} = \frac{\sigma - 1}{\epsilon_i - 1} \tilde{\beta}_i (\Psi_s - I) e_k.
\]

Putting this all into a matrix gives
\[
\frac{d\tilde{\beta}'}{d \log(M)} = \text{diag}(\tilde{\beta}) \text{diag} \left( \frac{\sigma - 1}{\epsilon - 1} \right) (\Psi_s - I).
\]

■

**Proof of lemma 5.3.** Recall that
\[
\tilde{\alpha} = (I - \tilde{M}^{\sigma - 1} \mu^{1 - \sigma} \Omega)^{-1} \tilde{M}^{\sigma - 1} \mu^{1 - \sigma} \alpha.
\]

To simplify the notation, for this proof, let
\[
B = (I - \tilde{M}^{\sigma - 1} \mu^{1 - \sigma} \Omega)^{-1}.
\]
So
\[
\frac{1}{M_i} \frac{d\tilde{\alpha}}{d \log(M_i)} = \frac{d\tilde{\alpha}}{dM_i'}
\]
\[
= B \frac{d\tilde{M}^{\sigma^{-1}}}{dM_i} \mu^{1-\sigma} \alpha + B \frac{d(\tilde{M}^{\sigma^{-1}})}{dM_i} (\tilde{M}^{\sigma^{-1}})^{-1} (B \tilde{M}^{\sigma^{-1}}) \mu^{1-\sigma} \Omega B (\tilde{M}^{\sigma^{-1}}) \mu^{1-\sigma} \alpha,
\]
\[
= B \frac{d\tilde{M}^{\sigma^{-1}}}{dM_i} \mu^{1-\sigma} \alpha + B \frac{d(\tilde{M}^{\sigma^{-1}})}{dM_i} (\tilde{M}^{\sigma^{-1}})^{-1} (B - I)(\tilde{M}^{\sigma^{-1}}) \mu^{1-\sigma} \alpha,
\]
\[
= B \frac{d(\tilde{M}^{\sigma^{-1}})}{dM_i} (\tilde{M}^{\sigma^{-1}})^{-1} B (\tilde{M}^{\sigma^{-1}}) \mu^{1-\sigma} \alpha,
\]
\[
= B \frac{d(\tilde{M}^{\sigma^{-1}})}{dM_i} (\tilde{M}^{\sigma^{-1}})^{-1} \tilde{\alpha},
\]
\[
= \Psi_d (\tilde{M})^{-\sigma} \mu^{1-\sigma} \frac{d(\tilde{M}^{\sigma^{-1}})}{dM_i} (\tilde{M}^{\sigma^{-1}})^{-1} \tilde{\alpha},
\]

The \( k \)th element of this vector is
\[
\frac{d\tilde{\alpha}_k}{d \log(M_i)} = \left( \frac{\sigma - 1}{\varepsilon_i - 1} \right) \left( \frac{\varepsilon_i}{\varepsilon_i - 1} \right)^{\alpha-1} e_k \Psi_d e_i \tilde{\alpha}_i \frac{1}{M_i'}.
\]

Putting this all into a matrix gives
\[
\frac{d\tilde{\alpha}}{dM} = \Psi_d \text{diag}(\tilde{\alpha}) \mu^{\sigma-1} \text{diag}((M)^{1-\sigma}) \text{diag}(\sigma - 1) \varepsilon - 1).
\]

Proof of proposition 5.4. Note that
\[
M_i = \frac{\bar{\beta}_i}{\varepsilon_if_i} P_i^C.
\]

Therefore,
\[
\frac{d \log(M)}{d\alpha_i} = M \frac{dP_i^C/\alpha_i}{P_i^C} + M \text{diag}(\bar{\alpha})^{-1} \frac{d\bar{\alpha}}{dM} \text{diag}(\bar{\alpha})^{-1} \frac{dM}{d\alpha_i} + M \text{diag}(\bar{\alpha})^{-1} \Psi d e_i.
\]

Rewrite this as
\[
\frac{d \log(M)}{d\alpha_i} = M \frac{dP_i^C/\alpha_i}{P_i^C} + M \left( \text{diag}(\bar{\alpha})^{-1} \Psi_1 + \text{diag}(\bar{\alpha})^{-1} \Psi_2 \right) \frac{dM}{d\alpha_i} + M \text{diag}(\bar{\alpha})^{-1} \Psi d e_i.
\]

Rearrange this to get the desired result.
Proof of proposition 5.6. Note that

\[
C = \frac{P_c C}{\bar{P}_c} = 1 + \sum_k \tau_k = \left( \frac{1}{\beta' \Psi_{d(\alpha \circ z^{\sigma-1})}} \right)^{\frac{1}{\tau_0}}.
\]

Therefore,

\[
\frac{d \log(C)}{dz_i} = \frac{1}{\sigma - 1} \frac{d \left( \log \left( \beta' \Psi_{d(\alpha \circ z^{\sigma-1})} \right) \right)}{dz_i},
\]

\[
= \frac{1}{\sigma - 1} \frac{1}{\beta' \bar{\alpha} \beta' \Psi_{d(\alpha \circ z^{\sigma-1})}} \frac{dM}{dz_i} + \beta' \Psi_{d(\alpha \circ z^{\sigma-1})} \frac{d}{dz_i},
\]

by lemma 5.3 and proposition 5.4,

\[
= \frac{1}{\sigma - 1} \frac{1}{\beta' \bar{\alpha} \beta' \Psi_{2MM^{-1}(I - \Lambda)^{-1}} \left( 1 + \frac{dP^\gamma_C/\bar{P}_c}{C} \right) \frac{d}{dz_i}} + \frac{1}{\beta' \bar{\alpha} \beta' \Psi_{d(\alpha \circ z^{\sigma-1})}} \frac{d}{dz_i}.
\]

Note that

\[
\frac{dP^\gamma_C/\bar{P}_c}{C} = \frac{d \log(P^\gamma_C)}{dz_i} = -(\sigma - 1) \frac{d \log(C)}{dz_i}.
\]

Substituting this into the previous expression and rearranging gives

\[
\left( 1 + \frac{\beta' \Psi_{2(I - \Lambda)^{-1}}}{\beta' \bar{\alpha}} \right) \frac{d \log(C)}{dz_i} = \frac{1}{\beta' \bar{\alpha} \beta' \Psi_{2(I - \Lambda)^{-1}} \left( \text{diag}(\bar{\alpha})^{-1} + I \right) \Psi_{d(\alpha \circ z^{\sigma-2})}}.
\]

Rearranging this gives the desired result.

\[\blacksquare\]

Proof of proposition 5.7. Let \(\bar{s}\) be the sales of the representative firm in each industry. Then

\[
\frac{d \log(\bar{s})}{d\alpha} = \frac{d}{d\bar{\alpha}} \left( \log(\bar{\beta}) + \log(\bar{\alpha}) - \log(M) \right) + \frac{d}{d\bar{\alpha}} \log(P^\gamma_C).
\]
By the chain rule,

\[
\frac{d \log(\tilde{\beta})}{d \alpha} = \left( \text{diag}(\tilde{\beta})^{-1} \frac{d \tilde{\beta}}{d M} + \text{diag}(\tilde{\alpha})^{-1} \frac{d \tilde{\alpha}}{d M} - \text{diag}(M)^{-1} \frac{d M}{d \alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{d} e_{k} + \frac{d}{d \alpha} \log(P_{c}^{\rho} C) \right)
\]

\[
= \left( \text{diag}(\tilde{\beta})^{-1} \Psi_{1}^{\prime} \text{diag}(M)^{-1} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{2} \text{diag}(M)^{-1} - \text{diag}(M)^{-1} \frac{d M}{d \alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{d} e_{k} + \frac{d}{d \alpha} \log(P_{c}^{\rho} C) \right)
\]

\[
= \left( \text{diag}(\tilde{\beta})^{-1} \Psi_{1}^{\prime} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{2} - I \right) \text{diag}(M)^{-1} \frac{d M}{d \alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{d} e_{k} + \frac{d}{d \alpha} \log(P_{c}^{\rho} C)
\]

\[
= \left( \text{diag}(\tilde{\beta})^{-1} \Psi_{1}^{\prime} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{2} - I \right) \text{diag}(M)^{-1} \frac{d M}{d \alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{d} e_{k} + \frac{d}{d \alpha} \log(P_{c}^{\rho} C),
\]

where the second to last line uses proposition 5.4.

**Proof of proposition 5.8.** Let \( s \) be the sales of the industries. Then

\[
\frac{d \log(s)}{d \alpha} = \frac{d}{d \alpha} \left( \log(\tilde{\beta}) + \log(\tilde{\alpha}) + \log(P_{c}^{\rho} C) \right).
\]

By the chain rule,

\[
\frac{d \log(s)}{d \alpha} = \left( \text{diag}(\tilde{\beta})^{-1} \frac{d \tilde{\beta}}{d M} + \text{diag}(\tilde{\alpha})^{-1} \frac{d \tilde{\alpha}}{d M} \right) \frac{d M}{d \alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{d} e_{k} + \frac{d}{d \alpha} \log(P_{c}^{\rho} C),
\]

\[
= \left( \text{diag}(\tilde{\beta})^{-1} \Psi_{1}^{\prime} \text{diag}(M)^{-1} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{2} \text{diag}(M)^{-1} - \text{diag}(M)^{-1} \frac{d M}{d \alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{d} e_{k} + \frac{d}{d \alpha} \log(P_{c}^{\rho} C) \right)
\]

\[
= \left( \Lambda \text{diag}(M)^{-1} \frac{d M}{d \alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{d} e_{k} + \frac{d}{d \alpha} \log(P_{c}^{\rho} C) \right),
\]

which gives the desired result. Note that the second to last line of the derivation uses proposition 5.4, and the last line uses the fact that

\[
\Lambda(I - \Lambda)^{-1} + I = (I - \Lambda)^{-1}.
\]

**Proof of lemma 6.1.** Element \((u, v)\) of both \(\Psi_{d}\) and \(\Psi_{s}\) are a weighted sum of the number of directed walks from \(u\) to \(v\). If \(u\) and \(v\) are not connected, then this is always equal to zero. Similarly, element \((v, u)\) of both \(\Psi_{s}\) and \(\Psi_{d}\) must also be equal to zero.

Element \((u, k)\) of both \(\Psi_{d}\) and \(\Psi_{s}\) are weighted sums of the number of directed walks from \(u\) to \(k\). If \(u\) is not connected to \(v\), then no walk from \(u\) to \(k\) goes through \(v\), therefore the \((u, k)\)th element

46
of \(\Psi_d\) and \(\Psi_s\) do not depend on \(v\). A similar argument implies that the \((k, u)\)th element of \(\Psi_s\) and \(\Psi_d\) also do not depend on \(u\).

Since \(\tilde{\beta}_u\) and \(\tilde{\alpha}_u\) are linear combinations of the \(u\)th column and row of \(\Psi_1\) and \(\Psi_2\), and since the \((u, v)\) and \((v, u)\)th elements of \(\Psi_d\) and \(\Psi_s\) are equal to zero, and the \((k, u)\) and \((u, k)\)th elements of \(\Psi_s\) and \(\Psi_d\) do not depend on \(v\), then \(\tilde{\beta}_u\) and \(\tilde{\alpha}_u\) also do not depend on \(v\). ■

**Proof of theorem 6.2.** Consider a firm \(i \in \mathcal{B}\). Suppose that the failure of \(v\) results in an equilibrium where firms in the set \(\mathcal{C}\) fail. If \(i\) is not connected to any firms in \(\mathcal{C}\), then by lemma 6.1, \(\tilde{\alpha}_i\) and \(\tilde{\beta}_i\) are the same regardless of whether or not \(v\) is rescued. Since the profits of firm \(i\) can be expressed as

\[
\pi_i = \text{const} \; P^e_c (\tilde{\alpha}_i \times \tilde{\beta}_i),
\]

if firm \(i\) prefers for \(v\) to be rescued, it must be that the exit of \(v\) causes \(P^e_c\) to fall. Note that exits can only cause \(P^e_c\) to increase, so if \(P^e_c\) falls, it must be that \(C\) has fallen. Since \(C\) is the utility of the household, it follows that rescuing \(v\) must be Pareto-efficient.

Now suppose that two firms \(i\) and \(j\) in \(\mathcal{B}\) disagree. In particular, \(i\) does not want \(v\) rescued but \(j\) does. Then it must be the case that \(P^e_c\) is not falling, but either of \(\tilde{\alpha}_j\) or \(\tilde{\beta}_j\) has fallen – this will occur if and only if some firm in \(\mathcal{C}\) is connected to firm \(j\). ■

**Proof of proposition 5.11.** By theorem 3.3, the profits of industry \(i\) are

\[
\pi_i = \frac{\beta_i \alpha_i}{\epsilon_i} P^e_c,
\]

when \(\Omega = 0\). Therefore

\[
\frac{d\pi_i}{dM_j} = \frac{\beta_i \tilde{\alpha}_i}{\epsilon_i} \frac{dP^e_c}{dM_j}.
\]

We can write

\[
P^e_c = \frac{1 - M' f}{\beta' \tilde{\alpha} - \text{diag}(\epsilon)^{-1} \beta' \tilde{\alpha}}.
\]

Hence,

\[
\frac{dP^e_c}{dM_j} = \frac{-f_i}{\beta' \tilde{\alpha} - \text{diag}(\epsilon)^{-1} \beta' \tilde{\alpha}} - \frac{1 - M' f}{\beta' \tilde{\alpha} - \text{diag}(\epsilon)^{-1} \beta' \tilde{\alpha}} \beta_i (1 - 1/\epsilon_i) \frac{\sigma - 1}{\epsilon_i - 1} M_i^{\sigma-1} \mu_i^{\sigma-1} \alpha_i.
\]

Since \(\sigma > 1\), both of these terms are negative and industries can only be enemies. ■

**Proof of proposition 5.13.** Let \(\pi(M)\) be the vector industrial profits with mass of entrants \(M\). Note that an equilibrium of the continuous limit corresponds to \(M(0)\) such that \(\pi(M(0)) = 0\). Let \(M(\Delta)\) correspond to the equilibrium mass of entrants for some \(\Delta \gg 0\). Then \(\pi(M(\Delta)) \geq 0\). By the
mean-value inequality

\[ \|\pi(M(\Delta))\| = \|\pi(M(\Delta)) - \pi(M(0))\| \leq \|D\pi(M^*)\|\|M(\Delta) - M(0)\|, \]

where \( M^* \) is in the convex hull of \( M(0) \) and \( M(\Delta) \). Rearrange this expression to get the desired result. ■