Optimal Social Security Policy in an Aging Economy with Supply Disturbance

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Abstract

Social security can serve as a tool to transfer wealth across generations and relieve the overaccumulation of capital. In this paper, I try to explore the capability of the social security system to deal with demographic and production shocks. Using a modified OLG model with uncertain life expectancy and production, I consider the optimum under different preferences of the government. I find that if the government is Utilitarian, the optimal PAYG contribution rate should be a positive concave function of capital labor ratio. If the government is Rawlsian, a small adjustment in PAYG contribution rate can make a significant improvement in the consumption of the worst-off generation.

1 Introduction

Technological breakthroughs in the medical industry and the continuing decrease in fertility rates have generated an astonishing aging trend. In 1950, the old-age dependency ratio of the United States was 0.12, while by 2010 this number rose to 0.20. As projected by the United Nations, the old-age dependency ratio of the United States will increase sharply to 0.33 by 2030. Many other countries share this astonishing trend, which causes concerns about its potential economic consequences of the increase in the old-age dependency ratio.

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1The population age 65 and over divided by population age 15-64.

Its important to understand the degree to which changes in the age structure, with fewer working-age people supporting a larger non-working population, will slow economic growth, exacerbate the government’s fiscal problems, or lower the quality of life for the elderly who depend on social security.

This paper tries to explore the capability of social security to act as an effective tool for the government in dealing with the ongoing demographic changes in an economy with uncertain productivity. There are generally two types of social security systems. The first type of social security system, typically called an unfunded system, serves as the tool for immediate transfer of wealth between older and younger generations. The younger generation contributes to the unfunded system via tax payments depending on the specified contribution rate. In contrast, the funded system sets up a fund account for every individual. The funded system represents a source of capital for production, a key factor for long-run economic growth. The unfunded system works better in providing insurance against generational shocks since it can be a flexible tool for allocating wealth across generations.

In the US, the relevant social security program is the Old-Age, Survivors, and Disability Insurance (OASDI) program, commonly known as the Social Security system. The tax payment is collected into the trust fund and distributed to the qualified individuals, such as retired or disabled people and their dependents. Benefits paid from the trust fund—the primary insurance amount (PIA)—are a fraction of the beneficiary’s average working-years income level, coupled with cost of living adjustments. Undistributed trust fund proceeds represent a source of national savings, which can be used for future social security payment. If the trust fund assets fall short of the required disbursements, this will necessitate a transfer of money from a general fund to complete the payment.

In 2011, total OASDI expenditures made up of 4.9% of the nation’s GDP. An additional $103 billion was transferred from the general fund of the Treasury to the OASDI trust fund account to compensate for the lower payroll tax rate in effect for 2011. This suggests that

\footnote{Source: Annual Statistical Supplement to the Social Security Bulletin, 2012, Social Security Administration.}
some pressure on social security payments is appearing. The problem is vital, especially in a post-recession period with uncertain prospect and the accumulating national debt. This paper may be unable to give a complete answer to this problem, but it aims to provide a simple framework to enrich the tools for analysis.

In light of that aim, I model a mixture of a PAYG and a fully-funded social security system in a closed, overlapping generations (OLG) economy. For those countries implementing such a mixed system, such as Australia, Denmark and Switzerland, my model may generate an analysis close to the reality. For those countries with only one type of system, we can set a special mixing parameter value that makes the mixed system degenerate as a single system.

An important policy variable facing governments is the choice of the “contribution rate”—the social security tax rate times the share of social security tax revenues going to the PAYG system. Determination of the optimal contribution rate depends on the nature of the government’s preferences. This paper will demonstrate that, if the government possesses the Utilitarian preferences (that is, it regards every generation as the same) and tries to maximize total utility across generations, the optimal PAYG contribution rate should have a positive and concave relationship with the capital-labor ratio, which is approximately what is observed in US data. If the government possesses the Rawlsian preferences (that is, it cares more about the worst-off generation), then there should be a reduction in PAYG contribution rate for the unlucky generation and a rise in PAYG contribution rate for the lucky generation following it in the case of a temporary bad production shock.

One limitation of this model is that, the absence of government bonds rules out the possibility that the government borrows from other countries. This restriction of a wealth transfer tool forces the government in the model to adjust the social security policy as the only means for transferring wealth across generations. Therefore, this restriction makes this model unsuitable for analysis of the effects of social security on the size of the national debt. An extension of this model that adds the government bond market might do a better job in
predicting the performance of government budget in response to the aging trend. However, the absence of government debt also starkly illuminates an important question: aside from borrowing from the future or from other countries to sustain the current level of consumption, what can a government do to improve the welfare of the whole society? This paper tries to address this question.

The paper is organized as follows. Section 2 reviews the relevant literature concerning about social security and its economic effect. Section 3 presents the model and the competitive equilibrium. The welfare analysis and optimal policy discussion is given in Section 4. Section 5 concludes.

2 Literature Review

How social security affects economic growth, savings or capital accumulation has been controversial for at least five decades. Following Diamond’s (1965) pivotal work, Barro (1974) considered the effects of government bonds in an OLG with physical capital. He concluded that if there is an operative intergenerational transfer, in the sense that there is positive bequest or gift across generations, then a marginal change in imposed intergenerational transfers, such as social security payments, has no real effect on aggregate demand or the interest rates. Barro (1974) also considered the risk characteristics of government debt and showed that when tax liabilities are subject to variability, and private insurance arrangements entail transaction costs, marginal changes in government debt can have real effects on savings and capitalization. In contrast to Barro, Feldstein (1974) analyzed the impact of social security on individuals’ decisions about savings and retirement and estimated that the introduction of social security depresses personal saving by 30-50%. However, Leimer and Lesnoy (1982) cast doubt on Feldstein’s estimation result with new evidence and claimed that historical data cannot support the conclusion that social security depresses savings. Bloom, Canning, Mansfield and Moore (2007) focused on the issue
of increased longevity and savings behavior. They found that an increase in longevity raises aggregate savings rates in countries with universal pension coverage and retirement incentives. This effect disappears in countries with PAYG system and high replacement rates.  

Another focus of the effects of social security on human capital accumulation and fertility choice emerged with the development of the endogenous growth model. The seminal work of Becker and Barro (1988) showed that a permanent increase in unfunded social security benefits reduces fertility and raises physical capital intensity through raising the bequest cost per child. Zhang (1995) analyzed the effects of social security on the steady-state growth rate of income in an endogenous growth model with endogenous fertility. Zhang showed that the unfunded system promotes economic growth by reducing fertility and increasing the human capital accumulation per capita; the funded system may depress economic growth, depending on whether the receipt from the funded system is independent of contributions to the funded system. Zhang and Zhang (2003) considered a mix of earnings-dependent and universal social security benefits and concluded that the effects of social security on economic growth resulting from variations in the benefit formula can be very substantial.

This paper tries to find the optimal PAYG contribution rate in an OLG economy, which is similar to the context of Diamond (1965), Barro (1974) as well as Zhang and Zhang (2003). There are two contributions in this paper. The first is to set up a framework for the analysis of the socially optimal PAYG contribution rate in response to technology and demographic shocks. The second is to provide a candidate list of optimal social security policy for different government goals represented by government preferences. Welfare analysis in the literature above always takes the form of social planner’s problem, as in Yew, Zhang (2009). In contrast I tend to limit the market power of the government by forcing it sets the policy given the competitive equilibrium. The complexity now lies in the variety

4The replacement rate: the proportion of an average worker’s wage that a pension plan replaces after retirement.
of government preferences rather than the optimization problem itself. Compared to the (discounted) Utilitarian preference, Rawlsian preference incorporates more flexibility in the process of policy making and is more practical. Numerical results of the optimal policy for the Rawlsian government are presented and it is shown that a small adjustment in the PAYG contribution rate can be very effective in improving the situation of the worst-off generation.

3 Model

3.1 Demographics

The economy is populated with overlapping generations of identical households. The number of households in generation $t$ is $N_t$ and grows with rate $n_t$. Every household lives for two periods and can only earn income in the first (young) period. The lifespan for each household is uncertain, with a probability $\pi_t \in [0, 1)$ of surviving into the old age. Using the law of large numbers, the total population in a certain date $t$ can be expressed as $M_t = N_t + \pi_{t-1}N_{t-1}$. For our purposes, it is useful to find the old-age dependency ratio (the ratio of the old-age population to the working-age population), which is an important index for the age structure of a society, especially when we consider the PAYG social security system. In terms of the PAYG system, a large old-age dependency ratio implies a large transfer from the working generation to the retired generation. Denoted by $d_t$, the old-age dependency ratio in date $t$ is

$$d_t = \frac{\pi_{t-1}N_{t-1}}{N_t} = \frac{\pi_{t-1}}{1 + n_{t-1}} \tag{1}$$

where the second equality comes from the fact that $N_t = N_{t-1}(1 + n_{t-1})$. Holding everything else equal, a fall in the birth rate or an increase in life expectancy raises the old-age depen-
dency ratio, illustrating two sources of the aging trend: the fall in fertility and the extension of life expectancy.

3.2 Social security system

Although in most countries only a funded or an unfunded system is implemented, in this paper I allow for some flexibility and model a mixture of funded and unfunded systems. Countries such as Australia, Denmark, Switzerland and Uruguay have implemented such a mixed system (Bloom, 2007).

In the first period of each generation, every household is taxed \( \tau_t \) share of its wage \( w_t \). To model such a mixed system, I split the tax payment and let \( \alpha_t \) be the fraction devoted to the PAYG system, with \( 1 - \alpha_t \) going the fully funded system. Define \( \rho_t = \alpha_t \tau_t \) to be the PAYG contribution rate. If we fix \( \tau_t \), then an increase in \( \rho_t \) is equivalent to an increase in \( \alpha_t \), meaning that some funded pension is flowing into the PAYG system. Each period the tax revenue of PAYG system is distributed to old people, with \( P_t \) being the PAYG benefit per household in date \( t \) (for generation \( t-1 \)).

\[
\alpha_t \tau_t w_t N_t = \pi_{t-1} N_{t-1} P_t
\]

(2)

Note that \( d_t = \frac{\pi_{t-1}}{1+n_{t-1}} \), we have

\[
P_t = \frac{\rho_t w_t}{d_t}
\]

(3)

Equation (3) states that per-household PAYG benefit depends on the current young generation’s contribution (\( \rho_t w_t \)), and the size of old population sharing the benefit (\( d_t \)). Therefore, when the society becomes older (\( d_t \) increases), there should be a higher contribution rate in order to sustain a certain level of PAYG benefit. 5

In addition to the PAYG benefit, there are two additional sources of income for old-

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5For the case of U.S., since the formula computing PIA is designed to benefit the low income worker, there is a minimum payment that must be made. That is, there is a positive lower bound of \( P_t \). This is also true for most countries.
generation households: proceeds from the funded system and accumulated private savings. The period $t$ contribution to the fully-funded system will be invested and paid in $t+1$ to surviving households. This fund is fully disbursed each period. In period $t$, each household can also save privately by buying the annuity, which also pays all the collected funds to those who survive into the second period. The household savings and the fully-funded investment have the same rate of return $R_{t+1}$. Therefore, the old-age income of generation $t$ is

$$I^o_t = P_{t+1} + \frac{R_{t+1}[s_t + (1 - \alpha_t)\tau_t w_t]}{\pi_t}$$ (4)

### 3.3 Production and technology shock

There is a set of competitive firms using a Cobb-Douglas production technology with a random shock in each period drawn from a random sample having mean $\mu$ and variance $\theta$.

$$Y_t = F(K_t, L_t) = \epsilon_t K_t^\gamma L_t^{1-\gamma}$$ (5)

A recession that lasts for almost 30 years or more seems really unrealistic. However, in order to incorporate potential economic recession in an OLG model, we have to make such a compromise. Another way is treat the shock not as one lasting for 30 years but as an aggregation of any bad shock happening in the period.\textsuperscript{6}

Each household is endowed with one unit of labor, which inelastically supplies in production. As a result, $N_t = L_t$. Let $k_t = \frac{K_t}{L_t}$ be the capital labor ratio, then CRS implies that the production per capita is given by

$$f(k_t) = \epsilon_t k_t^\gamma$$ (6)

\textsuperscript{6}For a better fit in of a random shock, one can consider implementing the perpetual youth model that treats each period usually as one year.
The factor markets of this economy are competitive, therefore the individual wage rate \( w_t \) and the rate of return to capital \( R_t \) are equal to the marginal productivity of labor and capital, respectively.

\[
w_t = f(k_t) - k_t f'(k_t) = (1 - \gamma) \epsilon_t k_t^\gamma
\]

\[
R_t = f'(k_t) = \gamma \epsilon_t k_t^{\gamma-1}
\]

\[7\]

\[8\]

3.4 Household Preferences

Each household has a utility function \( U_t(c^y_t, c^o_t) = \log c^y_t + \beta \log c^o_t \), where consumption per household in generation \( t \) when young and when old are \( c^y_t, c^o_t \), respectively. Note that the utility function is a random variable since the realization of \( c^o_t \) is contingent on living into the second period. But because the utility function is time separable, by the law of large numbers we can take the average and set the utility function of a household of generation \( t \) to be

\[
U_t(c^y_t, c^o_t) = \log c^y_t + \beta \pi_t \log c^o_t
\]

In addition to the uncertainty of life expectancy, the maximum level of \( c^o_t \) is affected by the random technology shock \( \epsilon_{t+1} \) through \( R_{t+1} \) and \( w_{t+1} \) in \( I^o_t \). Therefore, the household observes all the state variables in its first period \( (k_t, \rho_t, \rho_{t+1}, \epsilon_t, \pi_t, n_t) \), and uses the expectation of factor prices \( R_{t+1}, w_{t+1} \) to make the optimal consumption decisions.

3.5 Competitive Equilibrium

Households’ private savings and the funded contribution are rented as capital to producers at the end of the first period and yield returns at the beginning of the second period. Therefore, the total amount of fund that generation \( t \) provides should be equal to the total
capital demand of generation $t+1$:

$$
[s_{t} + (1 - \alpha_{t})\tau_{t}w_{t}]N_{t} = K_{t+1}
$$

(10)

$$
k_{t+1} = \frac{s_{t} + (1 - \alpha_{t})\tau_{t}w_{t}}{1 + n_{t}}
$$

(11)

Define $\sigma_{t} = s_{t} + (1 - \alpha_{t})\tau_{t}w_{t}$ as the social savings of generation $t$. Given the expected factor prices $w_{t+1}$ and $R_{t+1}$, a household in generation $t$ maximizes its expected utility by solving

$$
\max_{c_{y_{t}}, c_{o_{t}}, s_{t}} \log c_{y_{t}} + \pi_{t}\beta E_{\epsilon}\{\log c_{o_{t}}\}
$$

with respect to

$$
c_{y_{t}} + s_{t} \leq w_{t}(1 - \tau_{t})
$$

(12)

$$
c_{o_{t}} \leq P_{t+1} + \frac{R_{t+1}(s_{t} + (1 - \alpha_{t})\tau_{t}w_{t})}{\pi_{t}}
$$

(13)

Since the objective function is increasing in $c_{y_{t}}$ and $c_{o_{t}}$, both constraints are binding. The Euler equation is

$$
\frac{1}{c_{y_{t}}^{\gamma}} = \beta E_{\epsilon}\{R_{t+1} \frac{1}{c_{o_{t}}^{\gamma}}\}
$$

(14)

Combine Equation(14) with the budget constraint (Equation(13)) and factor market equilibria (Equations(7) and (8)), we get the competitive equilibrium

$$
k_{t+1} = \frac{\beta \pi_{t}(1 - \rho_{t})(1 - \gamma)}{(1 + n_{t})(1 + \beta \pi_{t} + \frac{1 - \gamma}{\gamma} \rho_{t+1})} \epsilon_{t}k_{t}^{\gamma}
$$

(15)

$$
(\sigma_{t})^{*} = \frac{\beta \pi_{t}(1 - \rho_{t})(1 - \gamma)}{1 + \beta \pi_{t} + \frac{1 - \gamma}{\gamma} \rho_{t+1}} \epsilon_{t}k_{t}^{\gamma}
$$

(16)

$$
(c_{y_{t}})^{*} = \frac{(1 - \gamma)(1 - \rho_{t})(1 + \frac{1 - \gamma}{\gamma} \rho_{t+1})}{1 + \beta \pi_{t} + \frac{1 - \gamma}{\gamma} \rho_{t+1}} \epsilon_{t}k_{t}^{\gamma}
$$

(17)
\[(\epsilon_t^*) = (1 + \frac{1 - \gamma}{\gamma} \rho_{t+1}) \gamma d_{t+1} \epsilon_{t+1} k_{t+1}^{\gamma} \]  \hspace{1cm} (18)

The indirect utility of generation \(t\) is

\[v_t(\epsilon_{t+1}, k_{t+1}, \pi_t, n_t) = (1 + \beta \pi_t) \log (1 + \frac{1 - \gamma}{\gamma} \rho_{t+1}) + (1 + \beta \gamma \pi_t) \log k_{t+1} + \beta \pi_t \log \epsilon_{t+1} + \{\beta \pi_t \log \gamma d_{t+1} - \log \beta d_{t+1}\} \]  \hspace{1cm} (19)

Note that when there is no change in technology shock and demographics, and when the policy is stationary, then there is a unique equilibrium path leading to a unique steady state. Let \(k^*\) be the capital stock in steady state. Then we have

\[k^* = \left( \frac{\epsilon \beta \pi (1 - \rho)(1 - \gamma)}{(1 + n)(1 + \beta \pi + \frac{1 - \gamma}{\gamma} \rho)} \right)^{\frac{1}{1 - \gamma}} \]  \hspace{1cm} (20)

**Proposition 1**

1) An increase in longevity in period \(t\) will always have a positive effect on capital accumulation and savings. Specifically, if we rewrite the capital dynamics as

\[k_{t+1} = \phi(\pi_t, \rho_t, \rho_{t+1}) \epsilon_t k_t^{\gamma}, \]  

and savings as \((\sigma_t)^* = (1 + n_t)\phi(\pi_t, \rho_t, \rho_{t+1}) \epsilon_t k_t^{\gamma}, \) where

\[\phi(\pi_t, \rho_t, \rho_{t+1}) = \frac{\beta \pi_t (1 - \rho_t)(1 - \gamma)}{(1 + n_t)(1 + \beta \pi_t + \frac{1 - \gamma}{\gamma} \rho_{t+1})} \]  \hspace{1cm} (21)

then \(\frac{\partial \phi(\pi_t, \rho_t, \rho_{t+1})}{\partial \pi_t} > 0.\)

2) The introduction of PAYG social security has a negative effect on capital accumulation and savings, \(\frac{\partial \phi(\pi_t, \rho)}{\partial \rho} < 0.\)

**Proof.** See Appendix 1. ■

The intuition behind proposition 1 is clear. As people live longer, they become more patient and care more about the consumption in the future. The effect of PAYG social security system conforms with Feldstein's result that the introduction of social security depresses savings. In numerical results, given parameters \(\{n_t = 0, \pi_t = 0.9, \beta = 0.95, \gamma = 0.4, \epsilon_t = 1\},\) the steady state capital accumulation suffers a 26% decrease when \(\rho\) shifts from zero to 0.1.
In the long run, PAYG system depresses savings and economic growth; some criticisms also regard it as a Ponzi game program. However, in contrast to these criticisms and bad effects in theory, what we have seen is that PAYG system is heavily implemented around the world. The reason might be that this form of immediate transfer serves as an effective short-run policy tool to smooth consumption between young and old. Therefore, the government prefers using social security transfer to maintain political stability. But what is the optimal social security policy and how effective could it be in face of bad shocks? These will be answered in the next section.

4 Determination of optimal policy

4.1 Social preference

In this section, different types of social welfare functions are introduced in order to represent different possible preferences of the government. For instance, the government may care more about the welfare of old people, or the government may care more about the welfare of the generation that suffers an economic recession. Based on the choice of social welfare function, I will discuss the optimal social security policy \( \{\alpha_t, \tau_t\}_{t=0}^{\infty}. \)

There are two “sharp” social welfare functions: pure Utilitarian and Rawlsian. Pure Utilitarian represents an extreme stance of equality: everyone is equal. In terms of economics, this means different generations within a society have the same weight when the government assesses the welfare of them as a whole. However, because everyone has the same weight, the social preference is neutral to the unequal distribution of utility. On the contrary, Rawlsian preference represents another extreme—the government only cares about the worst-off one (the generation that encounters a bad production shock, for example). For this reason, the

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7When solving the optimum, I will rely on the single policy variable \( \rho \), which is the PAYG contribution rate. Since \( \rho = \alpha \tau \), when the optimum of \( \rho \) is derived, we will be able to discuss the optimum of \( \{\alpha_t, \tau_t\}_{t=0}^{\infty} \). For example, when the optimal adjustment for \( \rho \) is to decrease, the government can fix \( \tau \) and reduce \( \alpha \), or fix \( \alpha \) and reduce \( \tau \), or reduce both of them. In practice, because consumers are generally more sensitive to the changes in tax rate, fixing \( \tau \) and reducing \( \alpha \) might be a preferable choice for the government.
Rawlsian approach is also known as maxmin.

A difficulty in discussing the optimal social choice in OLG model is the infinity of generations. It is obvious that simply maximizing the pure Utilitarian social welfare function is not mathematically well-defined as long as infinitely many generations have positive utility (consumption). Compared to the Utilitarian approach, Rawlsian approach encounters fewer problems, as long as we can identify the worst-off generation among the infinite alternatives. However, Koopmans (1960, 1972) argued that discounted utility is sufficient for a well-defined social preference with infinite generations. Discounting the future is a popular assumption on the utility in economics. Intuitively it is also easy to defend such an approach. For a finitely-lived government, the welfare of the current citizens definitely outweighs the welfare of generations in the distant future. Technically, assume \( v_t \) is the indirect utility of generation \( t \) derived from competitive equilibrium, and let \( v = (v_1, v_2, ...) \) we can define the Utilitarian social welfare function as

\[
W_U(v) = (1 - \beta) \sum_{t=0}^{\infty} \beta^t v_t
\]  

Let \( \lambda_t = (1 - \beta) \beta^t \), we have \( \sum_{t=0}^{\infty} \lambda_t = 1 \), implying that \( \lambda_t \) is just the weight the government assigns to generation \( t \). We define a Rawlsian social welfare function as

\[
W_R(v) = \inf(v) = \inf\{v_1, v_2, ..., v_t, ...\}
\]  

The problem of finding the infinimum of an infinite sequence of indirect utilities is also challenging. Moreover, it is innocuous to focus on some finite sequence of shocks when we are interested in short-run analysis. Therefore, to discuss the optimum under Rawlsian preferences, I will specify some special cases, which are discussed in Section 4.3.
4.2 Social optimum under Utilitarian preference

Suppose first that the government possesses Utilitarian preferences and discounts the future. Then the government chooses the full contingent plan by solving the following optimization problem

$$\max_{\{\rho_t\}_{t=0}^{\infty}} (1 - \beta) \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{t+1} \{v_t\}$$

s.t. \(0 \leq \rho_t \leq 1\)

\(k_{t+1} = \phi(\pi_t, \rho_t, \rho_{t+1})\epsilon_t k_t^\gamma\)

\(\forall t = 0, 1, 2, ...\)

This is similar to the classic infinite horizon consumption planning problem. Before deriving the optimal condition for this problem, let us return to the indirect utility of generation \(t\), which was Equation (19) and is reproduced here as Equation (24):

$$v_t(\epsilon_{t+1}, k_{t+1}, \pi_t, d_t) = (1 + \beta \pi_t) \log(1 + \frac{1 - \gamma}{\gamma} \rho_{t+1}) + (1 + \beta \gamma \pi_t) \log k_{t+1}$$

$$+ \beta \pi_t \log \epsilon_{t+1} + \{\beta \pi_t \log \gamma d_{t+1} - \log \beta d_{t+1}\}$$

(24)

Rewrite it as a function of \(\{\epsilon_{t+1}, \epsilon_t, k_t, \pi_t, n_t, \rho_t, \rho_{t+1}\}\) by expanding \(k_{t+1}\):

$$v_t(\epsilon_t, \epsilon_{t+1}, \pi_t, d_t, \rho_t, \rho_{t+1})$$

$$= (1 + \beta \gamma \pi_t) \log(1 - \rho_t)$$

$$+ (1 + \beta \pi_t) \log(1 + \frac{1 - \gamma}{\gamma} \rho_{t+1}) - (1 + \beta \gamma \pi_t) \log(1 + \beta \pi_t + \frac{1 - \gamma}{\gamma} \rho_{t+1})$$

$$+ \gamma(1 + \beta \gamma \pi_t) \log k_t$$

$$+ \beta \pi_t \log \epsilon_{t+1} + (1 + \beta \gamma \pi_t) \log \epsilon_t$$

$$+ (1 + \gamma) \beta \pi_t \log d_{t+1} + \beta \pi_t \gamma \log \beta + (1 + \beta \gamma \pi_t) \log(1 - \gamma) + \beta \pi_t \log \gamma$$

policy term \(\rho_t\)

policy term \(\rho_{t+1}\)

capital term \(k_t\)

technology shock \(\epsilon_t, \epsilon_{t+1}\)

demographic term
The social security policy $\rho_{t+1}$ affects the utility of generations $t$ and $t+1$. An increase in $\rho_{t+1}$ reduces the utility of generation $t+1$ but increases the utility of generation $t$, that is, there is a trade-off between the welfare of generation $t$ and generation $t+1$. Therefore, the optimal social security policy should be the one that balances this trade off. Following this logic, I propose the following condition as the Euler equation:

$$\mathbb{E}\{|\frac{\partial v_t}{\partial \rho_{t+1}}|\} = \mathbb{E}\{|\beta R_{t+1}\frac{\partial v_{t+1}}{\partial \rho_{t+1}}|\}$$ \hspace{1cm} (25)

Namely, the marginal benefit for generation $t$ should equal to the marginal cost for generation $t+1$. For the moment, assume the constraint $0 \leq \rho_t \leq 1$ is unbinding. The policy function should be the one that solves the following functional equation

$$(1-\gamma)^{2-\gamma}(\frac{1 + \beta \pi_t}{1 + \gamma_{t+1} \rho_{t+1}} - \frac{1 + \beta \gamma \pi_t}{1 + \beta \pi_t + \gamma \rho_{t+1}}) = \frac{\mu(1 + \beta \pi_t \gamma)\epsilon_{t}^{-1}}{(1 - \rho_t)^{1-\gamma} k_{t}^{\gamma(1-\gamma)} d_{t+1}^{1-\gamma}} \cdot \frac{(1 + \beta \pi_t + \gamma \rho_{t+1})^{1-\gamma}}{(1 - \rho_{t+1})}$$ \hspace{1cm} (26)

where $\gamma = (1-\gamma)/\gamma$. It can be proved that there exists a unique solution to this equation, given state variables $\{\rho_t, \pi_t, \pi_{t+1}, n_t, \epsilon_t, k_t\}$:

**Proposition 2** Given state variables $\{\rho_t, \pi_t, \pi_{t+1}, n_t, \epsilon_t, k_t\}$, there is a unique $\rho_{t+1}$ that solves the equation.

**Proof.** Given state variables, the LHS of Equation (26) is decreasing in $\rho_{t+1}$ while the RHS of Equation (26) is increasing in $\rho_{t+1}$. Therefore, there is a unique solution to this equation. \hfill \square

Solving the closed form policy function $\rho_{t+1}(\rho_t, k_t)$ is quite complicated. I prefer to give a qualitative analysis using numerical method. Figure 1.A presents the graph of LHS of Equation (26) with respect to $\rho_{t+1}$ which is the black line, and the graph of RHS of Equation (26) with respect to $\rho_{t+1}$, which is the dash line. $\rho_{t+1}^{*}$ is pinned down at the horizontal
When there are no changes in technology or demographic shocks, we can determine the steady state optimal $\rho^*$ numerically, as displayed in figure 2. As is shown in the graph, the optimal PAYG share, which is pinned down at the intersection of two curves, lies somewhere between 10% and 20%. When the life expectancy increases, $\rho^*$ rises, which implies that when the government has Utilitarian preferences, it cares more about the old people.

When there are random shocks to technology or demographics, the steady state will vary with changes in state variables. Therefore, it is worth identifying how $\rho_{t+1}$ responses to these changes in state variables. Since we know the equilibrium path of capital $k_t$, it suffices to know the relationship between $\rho_{t+1}$ and $k_t$. Figure 1.B shows that there is a positive relationship between $\rho_{t+1}$ and $k_t$. The sequence of dash curves is drawn assuming different values of $k_t$. The solid red curve represents $k_t = 0.05$, while the solid blue one represents the highest value of $k_t$, which is 1. As $k_t$ increases from 0.05 to 1 by 0.05 each time, the dashed curve shifts to the right. The graph also shows that $\rho_{t+1}$ is concave in $k_t$, since the distance

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8 Figure 1.A is under parameter values $\{\beta = 0.9, \gamma = 0.3, \pi_t = 0.7, \pi_{t+1} = 0.7, n_t = 0, \epsilon_t = 1, \mu = 1, k_t = k^*, \rho_t = 0.1\}$. 
of shifting decreases as \( k_t \) becomes larger. If we define \( \rho_{t+1} = f(k_t) \), then \( f' > 0 \) and \( f'' < 0 \).

Now we get a qualitative trend of the optimal PAYG policy under Utilitarian preferences, which is summarized in the following propositions.

**Proposition 3** The optimal path of \( \rho_{t+1} \) fluctuates with the random shocks \( \epsilon_t \). Specifically, \( \rho_{t+1} \) is an increasing and concave function of \( k_t \). Let \( \Delta_{\rho}(k_t) = \rho_{t+1}(k_t) - \rho_t \), we have \( \Delta_{\rho}'(k_t) < 0 \).

**Proposition 4** Suppose the government has Utilitarian preferences and that there are no shocks:

1. If the initial capital \( k_0 \) is lower than the steady state, the capital increases along the optimal accumulation path. As capital accumulates, the optimal \( \rho_t \) increases as well. Let \( \Delta_{\rho}(t) \) denote the adjustment in policy, \( \Delta_{\rho}'(t) < 0 \). Finally, \( \rho_t \to \rho^* \) as \( k_t \to k^* \).

2. Conversely, if the initial capital per household \( k_0 \) is higher than the steady state, the capital per household decreases along the optimal accumulation path. As capital shrinks, the
optimal \( \rho_t \) decreases as well. Let \( \Delta \rho(t) \) denote the adjustment in policy, \( \Delta'_\rho(t) > 0 \). Also, \( \rho_t \to \rho^* \) as \( k_t \to k^* \).

4.3 Social optimum under Rawlsian preference

The analysis under Rawlsian preferences is more straightforward. The government cares more about the worst-off generation and spreads the bad shock for the worst-off generation to other generations. The government is said to possess the strict Rawlsian preferences if there is no weight for generations other than the worst-off generation. In the case of strict Rawlsian preferences, the government should make a policy that equalizes the consumption level of all generations following the worst-off generation (the random shock is unpredictable). However, in practice, no one can find a policy that lasts for several decades or even more. Therefore, in the modified Rawlsian preferences, or quasi-Rawlsian preferences, the government will only spread the bad shock to one or two other generations. Moreover, the government won’t cause too much loss for the “lucky” generations. In this section, three cases of shock and policy adjustment will be discussed. Because of the complexity of formulas and equations, I will rely on numerical examples.

4.3.1 Case 1. A Temporary downturn in productivity

Let us consider the case of a temporary production downturn. As introduced in the model setup, this downturn may come from a financial crisis, a war, or anything that might reduce productivity. The downturn might not last for a whole generation. What we are concerned about is the aggregate effects on the income of the generation subjected to the negative productivity shock. To measure welfare, note that the utility function is logarithmic form, I use the flat consumption equivalence (FCE) so as to avoid discussing negative values. Suppose there is a temporary bad shock only at time 12 (\( \epsilon_t = 1, t \neq 12 \)). Under parameter values \( \{ \beta = 0.95, \gamma = 0.4, n_t = 0, \pi_t = 0.5 \} \), I show the numerical result in figure 3. The red, dotted curve is the path of the flat consumption equivalence without a policy adjustment.
while the blue curve is the path with a policy adjustment.

As is shown in the graph, both generation 11 and generation 12 are affected once the downturn happens. The life income of generation 12 decreases as a result of the downturn when they are young. At the same time, generation 11 suffers because generation 12 is unable to support itself at a prior generation’s consumption levels. To improve the situation of generation 12, since \( v_t \) is increasing in \( \rho_{t+1} \) and decreasing in \( \rho_t \), it is natural to propose a policy change that reduces \( \rho_{12} \) and increases \( \rho_{13} \). The original policy is \( \rho_t = 0.2, \forall t \), while the government now sets \( \rho_{12} = 0.165 \) and \( \rho_{13} = 0.225 \). The policy adjustment achieves a 5.74% compensation in FCE for generation 11, while only sacrificing 1.69% and 1.46% of FCE for generation 11 and 13, respectively. Therefore, the adjustment spreads the shock to generation 12 forward and backward and achieves a significant compensation, without generating a large burden to other generations. The welfare of the whole society (across generations) thus improves. As we shall guess, a larger adjustment may generate a more smoothing path, but may levy too large a burden on other generations. However, the more generations we spread the shock to, the smaller the impact of the spread on other generations. So we combine these two and improve the adjustment, with \( \rho_{12} = 0.15 \), \( \rho_{13} = 0.225 \) and \( \rho_{14} = 0.21 \), which is depicted in the orange curve. Although not every generation benefits from this change in policy adjustment, the consumption path becomes smoother.

4.3.2 Case 2. A Temporary fall in the fertility rate

The fertility rate is one important factor that affects the optimal social security policy. As shown in section 3.1, the fall in fertility rate increases the old age dependency ratio thus generating upward pressure on the contribution rate. A falling fertility rate has been a trend worldwide, with developed countries having zero or even negative population growth rate. In the experiment, the initial population growth rate is 0, while in period 11 there is one-time

\[9\text{Suppose the FCE vector for period 11, 12, 13 is } \bar{c} = (\bar{c}_{11}, \bar{c}_{12}, \bar{c}_{13}). \text{ Then the vector without adjustment is } \bar{c} = (0.0756, 0.0678, 0.0756). \text{ The vector with policy adjustment is } \bar{c} = (0.0743, 0.0717, 0.0745).\]
decrease in the fertility rate to -0.2. The cause of such an astonishing fall in population growth rate can be institutional intervention, such as one-child policy in China, or a long-lasting war, such as WWII. Under parameter values \( \{\beta = 0.95, \gamma = 0.4, \epsilon_t = 1, \pi_t = 0.5\} \), I graph the simulation result in Figure 4.

The sharp decrease in the fertility rate at time 11 raises capital per household for generation 11 and the following generations. The welfare of generation 10 remains unaffected since the upward pressure of the contribution rate has been absorbed by the increment of wage. But there might exist some room for the government to smooth this jump in welfare. By setting \( \rho_{11} = 0.21 \), and \( \rho_{10} = 0.19 \), the government can smooth the path and improve the welfare of generation 10 without hurting generation 9 and 11 too much. However, such a policy adjustment is not feasible (an adjustment in period 10 when the shock is in period 11) if the fall in fertility is unforeseeable, such as a war.

### 4.3.3 Case 3. A Permanent increase in longevity

Increased longevity is another trend worldwide. From proposition 1, a rise in life expectancy has a positive effect on capital accumulation and savings, thus a positive effect on total income and consumption. However, this rise will also generate upward pressure on the
contribution rate. To see this, let us take a look at the case for the U.S. I use the old-age dependency ratio of the U.S. to capture the changes in \( \pi_t \) as the fertility rate has remained constant during the same period. I graph the numerical result for this case with the parameter values \( \{ \beta = 0.95, \gamma = 0.4, n_t = 0, \epsilon_t = 1 \} \), except that \( \pi_t = 0.2 \) when \( t \leq 11 \) and \( \pi_t = 0.29 \) when \( t \geq 12 \). Figure 5 shows that generation 11 suffers a decrease in consumption, and the generations following it are all better off. The intuition is that as the old age dependency ratio jumps from 0.2 to 0.29, there are too many old people to support for generation 12. The negative effect of this increase in longevity outweighs the benefit from it. Without any adjustment in social security policy, the adverse effect will be absorbed solely by generation 11. As with Case 2, I raise the contribution rate for generation 12 to 0.22 and reduce the one for generation 11 to 0.19, which generates a smoother path. Because the increase in longevity is foreseeable, there is no issue of political feasibility in this case.

5 Conclusion

In this paper, I try to explore the capability of the social security system to deal with demographic and productivity shocks. From the analysis above we can conclude that if the government has Utilitarian preferences, the optimal PAYG contribution rate should be a
positive concave function of capital-labor ratio. If the government has Rawlsian preferences, a small adjustment in the PAYG contribution rate can make a significant improvement in the consumption of the worst-off generation. Currently the US faces a period of a rising old-age dependency ratio and a sluggish growth. Both of these generate upward pressure on the contribution rate for the current young (for a constant $P_t$, we have to increase $\rho_t$ when $d_t$ goes up and $w_t$ drops down). However, according to the analysis above, there should be a reduction in the contribution rate for the current young generation in response to these two shocks.\footnote{The current young generation can be regarded as generation 12 in Case 1, generation 11 in Case 3.} In the real world, the difference between the optimal low contribution rate and the commonly proposed high contribution rate is absorbed by the government deficit, or an outflow of capital from the general fund. Therefore, without harming the old generation in the period with negative shocks\footnote{Generation 11 in Case 1, for example}, financing the gap with a large government deficit seems to be the only way out. The problem may be even more serious given that the government in the real world is much more myopic than in the model. Thus the final point may be pessimistic: a generational disaster is uninsurable.
References


**Appendix**

**A1. Proof of proposition 1**

**Proof.** (1)

$$\frac{\partial \phi}{\partial \pi_t} = \frac{\beta(1 - \rho)(1 + n_t)(1 + \beta \pi_t + \hat{\gamma} \rho_{t+1}) - \beta(1 + n_t)\beta \pi_t(1 - \gamma)(1 - \rho_t)}{(1 + n_t)^2(1 + \beta \pi_t + \hat{\gamma} \rho_{t+1})^2}$$

$$= \frac{\beta(1 - \rho)(1 + n_t)(1 - \gamma)(1 + \hat{\gamma} \rho_{t+1})}{(1 + n_t)^2(1 + \beta \pi_t + \hat{\gamma} \rho_{t+1})^2} > 0$$

Thus the increase in $\pi_t$ will promote savings and capital accumulation.

(2) Suppose the policy is stationary, namely, $\rho_t = \rho$, $\forall t$,

$$\frac{\partial \phi(\pi_t, \rho)}{\partial \rho} = \frac{-\beta \pi_t(1 - \gamma)(1 + n_t)(1 + \beta \pi_t + \hat{\gamma} \rho) - \hat{\gamma}(1 + n_t)\beta \pi_t(1 - \rho)(1 - \gamma)}{(1 + n_t)^2(1 + \beta \pi_t + \hat{\gamma} \rho_{t+1})^2} < 0$$

Therefore, the introduction of PAYG social security system will depresses savings and capital accumulation. ■