The Rise of Software and Skill Demand Reversal*

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Abstract

The pattern of job polarization shows that the rise of high-skill occupations shrunk with the greater increase of low-skill occupations since the late 1990s. We document a simultaneous rise of software investment with the change in the pattern and explain both using a model of sorting and directed technical change. We first provide a new empirical support that high-skill occupations rely more on software and middle-skill occupations rely more on equipment. Because of the differences in the capital use, a decline in the price of equipment leads to a reduction in the employment share of middle-skill occupations when occupations are complementary. The shrink in the middle then reduces demand for equipment, resulting in more software innovation compared to equipment. The rise of software innovation, in turn, leads to the reversal in demand for skill. A quantitative analysis shows that the model explains about a half of the rise of software and skill demand reversal observed in data.

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1 Introduction

In the United States, the employment shares of high- and low-skill occupations have increased relative to middle-skill occupations, labeled as job polarization. Though many studies have analyzed the job polarization, there have been relatively fewer attentions on different patterns by decades. The rise of high- and low-skill occupations, however, have shown very distinctive patterns. The increase of high-skill occupations has stagnated while that of low-skill occupations has boosted since the late 1990s, which we refer to as skill demand reversal following Beaudry et al. (2016) (figure 2.1).

This paper provides a technology based explanation for the skill demand reversal. We first document that the skill demand reversal has accompanied with faster increases in software investment relative to equipment investment, which we call the rise of software (figure 2.3). We argue that this change in the composition of investment is closely related to the changes in the occupational structure, and further, provide an explanation for both skill demand reversal and the rise of software in a unified framework.

We start by showing meaningful connections between capital types and occupations from a new data combining two datasets – National Income and Product Account (NIPA) and O*NET Tools and Technology Database. The new dataset shows that average amount of investment on software and equipment by occupations indeed have a tight link with the task of each occupation. Software is used intensively by cognitive (high-skill) occupations, and equipment is used intensively by routine (middle-skill) occupations (figure 2.5).

The next question is which mechanism could explain the observed changes in the pattern of investment composition and skill demand. To answer, we provide a model of sorting and directed technical change. The model has two novel features. First, the model has both a sorting of heterogeneous workers into heterogeneous tasks and an endogenous evolution of technologies. This enables us to analyze the static and dynamic implications of interactions between technology and labor market simultaneously. Second, the technological changes in the model are augmented in different types of the capital, having a direct mapping to well-established data in the National Account. This feature makes a quantification of the model clear and straightforward.

In the model, individuals are heterogeneous in human capital and sort into a continuum of tasks. The tasks are different in how intensively they use software and equipment, where the intensities come from the data observation mentioned above. The equipment and software are composite of infinitesimal varieties, provided by innovators who are free to choose a type of capital to innovate.
After characterizing the equilibrium, we prove a series of comparative statics, predicting job polarization, the rise of software and skill demand reversal in response to one exogenous change: a decline in the price of equipment. More precisely, when different tasks are complementary in production, the decline in the price of equipment leads to job polarization in static equilibrium. Then the rise of software innovation follows in dynamics, which, in turn, causes skill demand reversal accordingly.

The intuition behind the comparative statics is as following. The decline in the price of equipment makes middle-skill occupations more productive, as they use equipment more intensively than others. When tasks are complementary, other tasks should follow the productive middle-skill occupations, causing factors flow out from middle-skill to high- and low-skill occupations, i.e., job polarization.

With shrinking employment share of middle-skill occupations, demand for equipment also falls as middle-skill occupations are using equipment more than others. Hence innovation toward software becomes more profitable, leading to the rise of software in dynamics. The rise of software then makes high-skill occupations more productive as they use software more than others. Following the same logic, demand for high-skill occupations falls, which is skill demand reversal.

We check an empirical validity of the mechanism in the model using an industrial variation. Using the fact that the decline in the relative price of equipment to software varies across industries, we test two predictions of the model. The model predicts a negative relationship between the speed of decline in the relative price of equipment to software and the growth of middle-skill employment relative to high-skill employment. It also predicts a positive correlation between the speed of decline in the relative price of equipment to software and the growth of software innovation to R&D other than software. We confirm significant correlations in both cases.

The last question is how important the mechanism is. We quantify the importance of the mechanism using the calibrated model, discretizing a continuum of tasks to 10 occupation categories. The quantitative analysis shows that the channel of directed technical change can account for about a half of the rise of software and skill demand reversal, former measured by the relative size of software investment to equipment investment and latter measured by deviation of actual series from the trend of the first decade.

The result has mainly two important implications. First, software and equipment capital measured in National Accounts can be a good proxy for the technological changes shaping the structure of labor market. Considering that the technological changes have significant impacts on many economic variables, a careful investigation
of the composition of capital investment can be fruitful in understanding economic phenomena other than job polarization.

Second, a technological change directly affecting a particular group of occupations could lead to other types of technological change affecting other occupations eventually. In particular, this paper shows that recent technical change started to reduce cognitive intensive occupations as long as occupations use the software. We do not analyze labor supply side explicitly but the fact that high-skill occupations demanded less would affect labor supply as it will change expected returns from skill formation, for example, through a decision of education.

**Related Literature**  That the job polarization is related to the increases in the productivity of middle skilled occupations, which are intensive in the routine task, has been well documented in the literature (e.g., Autor et al., 2006; Autor and Dorn, 2013; Goos et al., 2014, among others). Though not as many, some studies such as Beaudry et al. (2016) and Valletta (2016) have discussed the flattening demand for high skilled workers around the year 2000. This paper contributes to this literature by analyzing both polarization and skill demand reversal in a unified framework, and link the labor market phenomena to changes in the composition of capital investment.

Several papers analyze consequences of task specific technological changes on labor market using an assignment model (Costinot and Vogel, 2010; Lee and Shin, 2017; Stokey, 2016; Michelacci and Pijoan-Mas, 2016, among others). We also have an assignment feature similar to these works, but we characterize tasks by different use of two types of capital and introduce endogenous task specific technological change generated from innovation for each type of capital. By doing so, we can have a direct mapping of task specific technological change to observed data. In addition, we explain why a particular type of technology may or may not change.

Recent studies by Lee and Shin (2017) and Bárány and Siegel (forthcoming) show that either task specific technological change or sector specific technological change can lead to both job polarization and structural change. Since a single type of technological change can result in both phenomena, it is not easy to conclude whether the source of technological change has been task specific or sector specific. Our paper implies that the technological change embodied in a particular type of capital could be a source of task-specific technological change that can generate both phenomena.

Acemoglu and Restrepo (2016) and Hémos and Olsen (2016) also analyze interaction between technological change and labor market using directed technical change framework of Acemoglu (2002). While theirs give new insights on how automated
technology evolves and affects labor market outcomes, it is not straightforward to interpret the technology regarding observable data, especially in an aggregate sense. Our technological changes come from investment on software and equipment measured in National Accounts, and hence easier to interpret what they mean in data. Our tasks also have a clear interpretation as they are mapped directly to different occupations in data.

A seminal work by Krusell et al. (2000) also links changes in the price of equipment capital to skill-biased technical change to analyze the effects of technological change on labor market outcomes. They emphasize that skill-capital complementarity (capital substitutes low skill labor more than high skill labor) is a key to understanding how increasing productivity of capital induces increasing demand for high-skill workers. Different from Krusell et al. (2000), substitutability between labor and capital is same regardless of worker skills in our model. What is important is that various tasks use capital with different intensities, and the tasks are complementary.

The work by Krusell et al. (2000) and our paper, however, do not necessarily contradict each other as worker’s classifications are essentially different. They classify workers concerning education, and we classify workers regarding occupation. It seems natural that low educated workers could do what high educated workers usually do, while it should not be easy that certain occupations do what other occupations usually do. Indeed, recent papers such as Goos et al. (2014) and Lee and Shin (2017) also highlight complementarity between tasks as a key in understanding task level employment changes (i.e. polarization). In this regards, our paper complements Krusell et al. (2000) by linking the capital to task-based employment.

Another important feature of this paper is recognizing software capital distinguished from equipment capital. The software investment has increasing importance as its share in total investment is rapidly increasing. Koh et al. (2016), for example, emphasizes the importance of software capital (more broadly, Intellectual Property Products capital) in accounting for declining labor share in the US. While their analysis focuses on the relation between total labor and capital, we emphasize an important role of software investment in shaping the distribution of occupational demand. Though not of primary focus in this paper, we also observe declining labor share coming from software investment in the quantitative analysis and show that there indeed is a significant correlation between declining labor share and software intensities in industry level data.

We organize the rest of paper as follows. In section 2, we summarize the relevant empirical facts. In section 3, we present model and characterize the equilibrium. In
section 4, we conduct an analytical comparative statics based on the presented model. In section 5, we show a significant correlation consistent with the model’s prediction in industrial variation. In section 6, we quantify the importance of the model in accounting for the changes in employment structure from simulation exercise. Section 7 concludes.
2 Key facts

In this section, we document several data observations. First, the pattern of polarization has been changed since around the year 2000. Second, the composition of private investment also has been changed. Third, we show that middle skilled occupations use equipment intensively whereas high skilled occupations use software intensively. Moreover, the intensity of equipment and software across tasks are closely correlated with routine task intensity and cognitive task intensity.

2.1 The pattern of job polarization

Figure 2.1 shows the changes in the employment share across skill percentile by decades from 1980, computed from Census/ACS data. Each point in the skill percentile represents a group of occupations representing 1% of labor supply in 1980, sorted by average log hourly wage in 1979.

It is evident from the figure that there have been U-shaped changes in the employment share from 1980 to 2010. Looking at three lines separately, however, we see that the rise in high-skill occupations has been focused on the first decades whereas that of the low-skill occupations has boosted during 2000-2010. Moreover, the range of shrinking occupations is moving toward the right across decades.

We can see a similar observation by annual data from CPS. To see this, we classify occupations into three groups: cognitive (high-skill), routine (middle-skill), and low skill services (low-skill). Then we compare two different trends – linear trend from 1980-1995 and HP trend including all data points – of employment share of each occupational group. Figure 2.2 confirms that there have been breaks in the trends of employment shares of cognitive (high-skill) occupations and low skilled services (low-skill) occupations at around the late 1990s. Interestingly, the decline of routine (middle-skill) occupations has almost same trends before and after 1995.

2.2 The composition of investment

Now we turn to investment composition in the US. Were there any meaningful changes in the investment composition by capital type?

From NIPA, we compute share of software investment and equipment investment in total nonresidential investment and plot them in figure 2.3. Figure 2.3a shows

\[ \text{The classification of occupations is based on one digit SOC. The cognitive occupations are management, professionals, and technicians. The routine occupations are Machine operators, Transportation, Sales and office, Mechanics, and Miners and Production.} \]
increasing trend of software investment while figure 2.3b shows a decreasing trend of equipment investment. Moreover, the downward trend of equipment share has been faster since around mid-1990s.

We further decompose the share of equipment investment into sub items in equipment category. The investment on equipment in NIPA consists of four sub categories: Information Processing Products, Industrial equipment, Transportation equipment, and Other equipment. Figure 2.3c to 2.3f show the share of each equipment subitem in total non-residential investment. We can see that information processing and industrial equipment mostly drive the big drop in equipment investment around the late 1990s.

2.3 The decline in the price of capital

The natural question from the facts described so far is how could the changes in the pattern of polarization and investment composition be related, and what could explain the change. That we argue as an underlying factor causing two observations in subsection 2.1 and 2.2 is the decline of price of capital, which is often referred as investment-specific technological change in the literature (Greenwood et al., 1997; Cummins and Violante, 2002, for example).

Figure 2.4 shows the decline of the price of capital by investment sub item. We construct quality-adjusted price index for each investment item by extending Gordon (1990) measures of quality-bias in the official price indexes, following Cummins and
Fig. 2.2: Employment share of cognitive, routine, and low-skill services occupations

(a) Cognitive (high-skill)

(b) Routine (middle-skill)

(c) Low skill services (low-skill)

Note: 1) Cognitive occupations are management, professionals, and technicians. Routine occupations include office and sales, transportation, machine operators, mechanics, construction and production workers.
2) The blue line is linear trend of 1990 to 1995, and the red line is HP trend with smoothing parameter 100. All vertical axes represent 15%p of range.

Violante (2002). It is well-known that the price of capital shows declining trend compared to the price of consumption. What is less stressed in the literature is that the price of equipment decreases faster than that of software, especially from mid-1990’s.

Why could the decline in the price of equipment cause the changes in investment composition and demand for skills? Suppose for now that middle-skill occupations use equipment more and high-skill occupations use software more. Suppose also that innovators are free to choose the type of capital to innovate. The decline in the price of equipment makes the middle-skill occupation more productive, and so firms may need
Fig. 2.3: Investment share in private non-residential investment

(a) Software

(b) Total Equipment (A+B+C+D)

(c) Information Processing (A)

(d) Industrial Equipment (B)

(e) Transportation Equipment (C)

(f) Other Equipment (D)

Note: The red line shows HP-trend with smoothing parameter 100. All vertical lines are representing 20 %p of range.
Fig. 2.4: Log of quality-adjusted price index by capital type (1980=0)

Note: The quality-adjusted price index is constructed by extending Cummins and Violante (2002).

less of them. The demand for equipment could be lowered then, which makes innovation for software more profitable in innovators’ perspectives. New software products are introduced to the economy, making high-skill occupations more productive, which in turn, could lead to less hire of high-skill occupations by firms.

The detailed description and conditions of this potential mechanism are to be introduced in section 3 and 4 where we do analysis using a model of endogenous sorting and directed technical change. A question is then whether we can check by data if there is a meaningful connection between the use of equipment and software and occupations.

2.4 Use of Equipment and Software across occupations

We provide data evidence that document tight connections between the use of different types of capital across occupations. To be specific, we construct capital use by occupation data by combining two data sources – National Income and Product Account (NIPA) and O*NET Tools and Technology Database.

The O*NET Tools and Technology database provides information on types of tools and technology (software) that each occupation is using for its task. One caveat of this dataset is that this does not provide any information on the values of each item. To fill this caveat, we try to link capital items in O*NET Tools and Technology to NIPA data obtained from the Breau of Economic Analysis (BEA).

Specifically, we make a naive concordance between UN Standard Product and Services Code (UNSPSC), a product classification system used in O*NET database, and
25 categories of nonresidential equipment in NIPA table 5.5 (details in the appendix A). And then we distribute total amount of particular type of equipment investment to each occupation using the tools that are included in the investment category using the concordance.

For example, suppose that firms have invested USD 20 billion in metalworking machinery in NIPA table. According to the constructed concordance, the metalworking machinery includes total 139 commodities in UNSPSC. Some occupations use none of the 139 commodities, and some occupations use a certain number of commodities in the category. Since we know the number of employment by occupation, we can calculate the total number of items in metalworking machinery used by all workers in a given year. Now we could approximate an amount attributed to an individual occupation, by distributing total USD 20 billion according to the number of items used by the occupation. Dividing by the number of employees, we estimate per capita investment on the metalworking machinery by occupation.

The amount of per capita investment in equipment by occupational skill group is in figure 2.5a, where the occupational skill group is defined as a group representing 1% of employment among all the occupations ranked by mean hourly wages. We also plot the routine intensive task share – a share of the routine intensive employment out of total employment in the skill group – in the same figure. Here the routine intensive employment is defined as the employment of an occupation with the highest one-third routine task index out of all occupations, where the routine task index is computed using O*NET task database following Acemoglu and Autor (2011).

In figure 2.5b, we plot software investment per capital across the same wage percentile, and the cognitive intensive task share similarly defined as routine task intensive share. Again, cognitive task index is computed following Acemoglu and Autor (2011).

We can see from the figures that middle-skill workers use equipment more intensively, and the high-skill workers use software more intensively. Moreover, the use of equipment closely follows the routine task share while the use of the software is closely related to the cognitive task share. We further show the use of equipment subitem by occupation in figure 2.5c (industrial equipment) and 2.5d (industrial and information processing equipment). Among equipment subitems, the industrial equipment is one that is most correlated with routine task intensity.
Fig. 2.5: Use of equipment and software across skill percentile

(a) Total equipment

(b) Software

(c) Industrial equipment

(d) Industrial + Information processing equipment

Note: Detailed information on data is in appendix A.
3 Model

There is a continuum of individuals endowed with human capital \( h \in [1, \bar{h}] \) drawn from a measure \( \mathcal{M}(h) \). To be specific, we assume that:

**Assumption 1 (distribution)** The measure of skill, \( \mathcal{M} : [1, \bar{h}] \mapsto [0, 1] \) is a cumulative distribution function with a differentiable probability distribution function, \( \mu : [1, \bar{h}] \mapsto \mathbb{R}^+ \).

There is a continuum of task \( \tau \in [0, \bar{\tau}] \), and final goods are produced by combining task output \( T(\tau) \) according to:

\[
Y = \left( \int_{\tau} \gamma(\tau)^{1/2} T(\tau)^{\gamma-1} d\tau \right)^{\gamma/(\gamma-1)}.
\]  

(1)

The task output is produced by integrating human capital specific task production \( y(h, \tau) \) across all skill levels used for the production of task \( \tau \):

\[
T(\tau) = \int_{h \in \mathcal{L}(\tau)} y(h, \tau) dh.
\]  

(2)

The human capital specific task production, \( y(h, \tau) \), depends not only on worker’s human capital \( h \) but also on task \( \tau \) that the worker is working for. Specifically, the functional form of \( y(h, \tau) \) is given by

\[
y(h, \tau) = \left\{ \alpha_h(\tau) (b(h, \tau)l)^{\frac{\sigma_e-1}{\sigma_e}} + \alpha_e(\tau) E^{\frac{\sigma_e-1}{\sigma_e}} \right\}^{\frac{\sigma_e \sigma_s-1}{\sigma_s}} + \alpha_s(\tau) S^{\frac{\sigma_s-1}{\sigma_s}}
\]  

(3)

where \( l(h) \) refers to level of employment of workers with human capital \( h \). \( S \) and \( E \) refer to software and equipment, respectively.

The function \( b(h, \tau) \) captures the productivity of a worker with human capital \( h \), when she works for task \( \tau \). We assume that \( b(h, \tau) \) is strictly log supermodular.

**Assumption 2** The function \( b(h, \tau) : [1, \bar{h}] \times [0, \bar{\tau}] \mapsto \mathbb{R}^+ \) is differentiable and strictly log supermodular. That is,

\[
\log b(h', \tau') + \log b(h, \tau) > \log b(h, \tau') + \log b(h', \tau),
\]

for all \( h' > h \) and \( \tau' > \tau \).

As shown in Costinot and Vogel (2010), the assumption 2 would help ensure a positive assortative matching (PAM). In other words, the higher human capital \( h \) is, the higher \( \tau \) task she will work for in equilibrium.
Although a positive assortative matching is obtained from exogenous productivity term \( b(h, \tau) \), \( \tau \) is not merely an index for task. Note that the task production \( y(h, \tau) \) depends on \( \tau \) also through different intensities of two types of capital, \( \alpha_s(\tau) \) and \( \alpha_e(\tau) \).

We further assume that software and equipment available for workers are given by

\[
S = \left( \int_0^{N_s} s(k)^{\nu_s} dk \right)^{\frac{1}{\nu_s}} \quad \text{and} \quad E = \left( \int_0^{N_e} e(k)^{\nu_e} dk \right)^{\frac{1}{\nu_e}},
\]

where each variety of capital \( (s(k) \text{ and } e(k)) \) is provided by a permanent patent owner under monopolistic competition with constant the marginal cost of producing capital, \( q_s \) and \( q_e \).

New software and equipment are created from R&D expenditures \( Z_s \) and \( Z_e \), and the law of motions for total varieties follow

\[
\dot{N}_s = Z_s/\eta_s \quad \text{and} \quad \dot{N}_e = Z_e/\eta_e.
\]

Finally, the representative household has CRRA preference, given by

\[
\int_s^{\infty} e^{-\rho t} C(t)^{1-\theta} \frac{1}{1-\theta} dt,
\]

and the resource constraint in the economy is

\[
C + q_e \int_0^{N_s} s(k)dk + q_s \int_0^{N_e} e(k)dk + Z_e + Z_s \leq Y.
\]

3.1 Static Equilibrium

To characterize the static equilibrium, we take total varieties of software and equipment, \( N_s \) and \( N_e \), as given for now. We first define the equilibrium.

**Definition 1 (Static equilibrium)** The static equilibrium consists of the price function \( p(\tau) \), \( w(h) \), \( p_s(k) \), and \( p_e(k) \), the quantity function \( T(\tau) \), \( l(h, \tau) \), \( s(k, \tau) \), \( e(k, \tau) \), and the quantity \( Y \) such that

1. Given \( p(\tau) \), final goods producer solves

\[
\max \ Y - \int_\tau p(\tau)T(\tau)d\tau,
\]

given equation (1)
2. For each task, the task output is produced to solve

\[
\max p(\tau)T(\tau) - \int_{h} w(h)l(h, \tau)dh - \int_{0}^{N_s} p_s(k)s(k, \tau)dk - \int_{0}^{N_e} p_e(k)e(k, \tau)dk,
\]

given equation (2), \(w(h), p_s(k), \) and \(p_e(k).\)

3. A capital provider solves

\[
\max \pi_s(k) = \int_{\tau} [p_s(k)s(k, \tau) - q_s]d\tau,
\]
\[
\max \pi_e(k) = \int_{\tau} [p_e(k)e(k, \tau) - q_e]d\tau,
\]

given the marginal cost \(q_s,\) and \(q_e.\)

4. All workers choose the highest paying occupation (task).

5. The labor market clears: \(\mu(h) = \int_{\tau} l(h, \tau)d\tau.\)

From the final goods production, the demand for task output \(T(\tau)\) is given by

\[
p(\tau) = \left(\frac{\gamma(\tau)Y}{T(\tau)}\right)^{\frac{1}{\epsilon}}, \tag{7}
\]

and the price function \(p(\tau)\)'s satisfy \(\int_{\tau} \gamma(\tau)p(\tau)^{1-\epsilon}d\tau = 1.\)

Since we assume that the capital producer maximizes profit under the monopolistic competition, we get the price of software and equipment as

\[
p_s(k) = \frac{q_s}{\nu_s} \quad \text{and} \quad p_e(k) = \frac{q_e}{\nu_e}, \quad \text{for all} \quad k.
\]

Substituting this to the first order conditions from task output production, we can show that the wage function \(w(h)\) satisfies

\[
w(h) \geq \left\{ p(\tau)^{1-\sigma_s} - \left(\frac{\alpha_s(\tau)^{1-\sigma_s}q_s}{N_s^{\nu_s}\nu_s}\right)^{\frac{1-\sigma_s}{3-\sigma_s}} \right\} \times b(h, \tau),
\]
\[
:= \omega(\tau) \times \alpha_s(\tau)^{\frac{\sigma_s}{1-\sigma_s}}
\]

with equality when \(l(h, \tau) > 0.\)

The equation (8) shows that the wage function \(w(h)\) can be expressed as a product of term depending only on \(\tau (\omega(\tau))\) and human capital-task specific productivity \(b(h, \tau).\)

Now the existence of a positive assortative matching between \(h\) and \(\tau\) follows:
Lemma 1 (Positive assortative matching) Under assumption 1 and 2, there exists a continuous and strictly increasing assignment function \( \hat{h} : [0, \bar{\tau}] \mapsto [1, \bar{h}] \) such that \( \hat{h}(0) = 1 \) and \( \hat{h}(\bar{\tau}) = \bar{h} \).

The proof is same as the proof of Lemma 1 in Costinot and Vogel (2010) and omitted.

The equilibrium assignment \( \hat{h} \) is characterized by:

Lemma 2 (Equilibrium assignment function) The equilibrium assignment function \( \hat{h}(\tau) \), price function \( p(\tau) \), and the wage rate \( \omega(\tau) \) satisfy the following system of differential equations.

\[
\begin{align*}
\frac{d \log \omega(\tau)}{d\tau} &= -\frac{\partial \log b(\hat{h}(\tau), \tau)}{\partial \tau}, \\
\hat{h}'(\tau) &= \gamma(\tau)p(\tau)^{\sigma_s - \epsilon} \alpha_s(\tau)^{\sigma_s} \psi(\tau)^{\sigma_s - \sigma_s} Y, \\
p(\tau) &= \left[ \psi(\tau)^{1-\sigma_s} + \alpha_s(\tau)^{\sigma_s} \left(q_s N_s^{-\varphi_s}/\nu_s\right)^{1-\sigma_s} \right]^{\frac{1}{1-\sigma_s}},
\end{align*}
\]

with \( \hat{h}(0) = 1, \hat{h}(\bar{\tau}) = \bar{h}, \) and \( \int \gamma(\tau)p(\tau)^{1-\epsilon}d\tau = 1, \)

where \( \psi(\tau) := \left[ 1 + \alpha_s(\tau)^{\sigma_s} \left(q_s N_s^{-\varphi_s}/\nu_s\right)^{1-\sigma_s} \right]^{\frac{1}{1-\sigma_s}}, \varphi_s := \frac{1-\nu_s}{\nu_s}, \) and \( \varphi_e := \frac{1-\nu_e}{\nu_e}. \)

**Proof** In appendix C.

Once the assignment function \( \hat{h} \) is obtained, all the equilibrium quantities and prices can be computed accordingly.

### 3.2 Dynamic equilibrium

Now consider a dynamic equilibrium where technology evolves endogenously. The HJB equations for innovators are given by,

\[
\begin{align*}
\begin{align*}
\frac{r(t)}{\nu_s}V_s(k, t) - \dot{V}_s(k, t) &= \pi_s(k, t), \\
\frac{r(t)}{\nu_e}V_e(k, t) - \dot{V}_e(k, t) &= \pi_e(k, t),
\end{align*}
\end{align*}
\]

with the profit function,

\[
\begin{align*}
\pi_s(k) &= \int_{\tau} [p_s(k)s(k, \tau) - q_s s(k, \tau)]d\tau = \frac{1-\nu_s}{\nu_s} q_s \int_{\tau} s(k, \tau)d\tau, \\
\pi_e(k) &= \int_{\tau} [p_e(k)e(k, \tau) - q_e e(k, \tau)]d\tau = \frac{1-\nu_e}{\nu_e} q_e \int_{\tau} e(k, \tau)d\tau.
\end{align*}
\]

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Free entry condition ensures that

\[ V_e \leq \eta_e, \text{ with equality if } Z_e > 0, \text{ and} \]
\[ V_s \leq \eta_s, \text{ with equality if } Z_s > 0. \]

As long as both R&D’s are positive, we have \( \eta_e V_e = \eta_s V_s \), and from equation (12) and (13),

\[ r(t) = \pi_e(t)/\eta_e = \pi_s(t)/\eta_s. \] (16)

Finally from the household’s problem, we have a standard Euler equation:

\[ \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}, \] (17)

and the transversality condition:

\[ \lim_{t \to \infty} \left[ e^{-\int_0^t r(s) ds} (N_e(t)V_e(t) + N_s(t)V_s(t)) \right] = 0. \]

Now we have a characterization of the steady state equilibrium in the following lemma.

**Lemma 3 (Steady state equilibrium)** There exist \( \nu_e < 1 \) and \( \nu_s < 1 \) large enough that are compatible with unique steady state equilibrium, i.e.

\[ \frac{\pi_e}{\eta_e} = \frac{\pi_s}{\eta_s} = \rho, \] (18)

and every variable stays at constant level. Moreover, when \( \sigma_s = \sigma_e = 1 \),

\[ \max \left\{ \frac{1-\nu_s}{\nu_s} \frac{\alpha_s(\tau)}{\alpha_h(\tau)} + \frac{1-\nu_e}{\nu_e} \frac{\alpha_e(\tau)}{\alpha_h(\tau)} \right\} < 1 \text{ ensures existence of the steady state equilibrium.} \]

**Proof** In appendix C.

To see intuition behind the result, consider a simple case with \( \sigma_e = \sigma_s = 1 \), i.e. Cobb-Douglas task production. We then can express the task output \( T(\tau) \) as

\[ T(\tau) = p(\tau) \kappa(\tau) N_s \Psi_{es}(\tau) \left( \frac{N_e}{N_s} \right)^{\Psi_e(\tau)} B(\tau), \] (19)

where \( \kappa(\tau) := \left( \frac{\nu_e \alpha_s(\tau)}{\nu_s \alpha_h(\tau)} \right)^{\alpha_s(\tau)} \left( \frac{\nu_e \alpha_e(\tau)}{\nu_e \alpha_h(\tau)} \right)^{\alpha_e(\tau)} \), \( \Psi_{es}(\tau) := \frac{1-\nu_s}{\nu_s} \frac{\alpha_s(\tau)}{\alpha_h(\tau)} + \frac{1-\nu_e}{\nu_e} \frac{\alpha_e(\tau)}{\alpha_h(\tau)} \), \( \Psi_e(\tau) := \frac{1-\nu_e}{\nu_e} \frac{\alpha_e(\tau)}{\alpha_h(\tau)} \), and \( B(\tau) := b(\hat{h}(\tau), \tau) \mu(\hat{h}(\tau)) \hat{h}'(\tau) \) are introduced to simplify notation (detailed derivation is in appendix C).
In any equilibrium with positive R&D, no arbitrage condition holds and so $\pi_s/\eta_s = \pi_e/\eta_e$. In this simple case, profit functions satisfy

$$\pi_s = (1 - \nu_s) \int_\tau \frac{\alpha_s(\tau)p(\tau)T(\tau)}{N_s} d\tau, \quad (20)$$

$$\pi_e = (1 - \nu_e) \int_\tau \frac{\alpha_e(\tau)p(\tau)T(\tau)}{N_e} d\tau. \quad (21)$$

From equation (20) and (21), no arbitrage condition implies that $N_s/N_e$ should be constant whenever the interest rate, $\pi_s/\eta_s$ (or $\pi_e/\eta_e$), is constant. When the condition $\max \left\{ \frac{1-\nu_s}{\nu_s} \frac{\alpha_s(\tau)}{\alpha_h(\tau)} + \frac{1-\nu_e}{\nu_e} \frac{\alpha_e(\tau)}{\alpha_h(\tau)} \right\} < 1$ holds, the marginal rate of return on $N_s$ is decreasing for all tasks. Moreover, for all tasks, the rate of return goes to infinity when $N_s \to 0$, and goes to 0 when $N_s \to \infty$. Because all the functions in the static equilibrium is continuous and differentiable, the rate of return is also continuous on $N_s$, the existence of steady state equilibrium follows. Basically, the parametric restriction given here implies that returns from increasing variety is small enough that the rate of return is decreasing in total variety.

We only consider a case with no growth steady state as there would not be a standard balanced growth path when the task production is general CES function. Note that the source of growth (increasing variety) is a capital augmented technological change in our model. It is well-known that there would not exist a balanced growth path with a capital-augmented technical change if a production function is not Cobb-Douglas form (e.g. Grossman et al., 2017).

A detailed analysis on the transitional dynamics is not of the primary focus of this paper. Instead, we focus on the differences between static equilibrium (in the very short run where $N_s$ and $N_e$ are fixed) and the steady state (where $N_s$ and $N_e$ have converged so as to $\pi_s/\eta_s = \pi_e/\eta_e = \rho$) throughout the paper. We confirm numerically that the obtained steady state is saddle path stable in the quantitative analysis.

---

2In Cobb-Douglas task production case ($\sigma_s = \sigma_e = 1$), however, we could have sustained growth assuming a strictly positive population growth as in Jones (1995). We still have every task growing at different rates, so the most labor intensive task (the slowest growing task) would dominate the economy in the limit under complementarity between tasks ($\epsilon < 1$), which is similar to that in Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008).
4 Comparative Statics

For this section, we restrict our attention to the case with \( \sigma_e = \sigma_s = 1 \), \( \eta_e = \eta_s \) and \( \nu_e = \nu_s \) to get analytical comparative statics. To be specific, we assume:

**Assumption 3** The elasticity of substitution between labor and equipment and labor and software are one, i.e. \( \sigma_s = \sigma_e = 1 \). The individual task production function is then

\[
y(h, \tau) = (b(h, \tau)l(h))^{\alpha_h(\tau)} E^{\alpha_e(\tau)} S^{\alpha_s(\tau)}.
\]

Also, we put some structures on the intensity function \( \alpha_h(\tau) \), \( \alpha_e(\tau) \), and \( \alpha_s(\tau) \) to reflect the fact that high-skill workers use software intensively and middle-skill workers use equipment intensively, i.e.,

**Assumption 4 (intensities)** The function \( \alpha_h(\tau) : [0, \bar{\tau}] \rightarrow (0, 1] \), \( \alpha_s(\tau) : [0, \bar{\tau}] \rightarrow (0, 1] \), and \( \alpha_e(\tau) : [0, \bar{\tau}] \rightarrow (0, 1] \) satisfy following.

2.1 \( \alpha_s(\tau) \) is differentiable and increasing on \( [0, \bar{\tau}] \).

2.2 \( \alpha_e(\tau) \) is differentiable, increasing on \( [0, \tau_e] \) and decreasing on \( [\tau_e, \bar{\tau}] \).

2.3 \( \alpha_e(\tau_e) > \alpha_s(\tau_e) \), \( \alpha_s(\bar{\tau}) > \alpha_e(\bar{\tau}) \), and \( \alpha_e(0) = \alpha_s(0) \).

Now we show that the decline in the price of equipment (\( q_e \downarrow \)) leads to all three polarization, the rise of software and skill demand reversal when the tasks are complementary. More specifically, we focus on three main predictions of the model: (1) a polarization induced by the falling price of equipment in the static equilibrium, and (2) subsequent rise of software, and (3) decreasing demand for high skilled employment in the steady state.

**Job polarization**

First, we show the impacts of an decrease in the price of equipment (\( q_e \downarrow \)) on equilibrium assignment function \( \hat{h}(\tau) \) in the static equilibrium (i.e. when \( N_e \) and \( N_s \) are fixed). We consider \( q_{1e} > q_{2e} \) and denote the equilibrium assignment function corresponding to \( q_{1e} \) and \( q_{2e} \) as \( \hat{h}_1 \) and \( \hat{h}_2 \), respectively.

**Proposition 1 (Polarization)** Consider \( q_{1e} > q_{2e} \). Suppose \( \epsilon < 1 \) and assumption 1 to 4. With small enough \( \alpha'_h(\tau) \), we have \( \tau^* \in (0, \bar{\tau}) \) such that \( \hat{h}_1(\tau^*) = \hat{h}_2(\tau^*) \), \( \hat{h}_1(\tau) < \hat{h}_2(\tau) \) for \( \tau \in (0, \tau^*) \), and \( \hat{h}_1(\tau) > \hat{h}_2(\tau) \) for \( \tau \in (\tau^*, \bar{\tau}) \).

**Proof** In appendix C.
The proposition 1 says that there will be a shrinking employment of task around $\tau^*$ where corresponding equipment intensity $\alpha_e(\tau^*)$ is relatively higher than $\alpha_e(0)$ and $\alpha_e(\bar{\tau})$. Figure 4.1 illustrates the change in the assignment function with $q_{1e}$ (blue solid line) and $q_{2e} < q_{1e}$ (red dashed line). For given task $\tau \in [\tau^*-\epsilon, \tau^*+\epsilon]$, we can see that employment working for the tasks becomes smaller. This is because we have higher $\hat{h}_2(\tau)$ on the left side of $\tau^*$ and lower $\hat{h}_2(\tau)$ on the right side of $\tau^*$, and hence $\mathcal{M}(\hat{h}_2(\tau^*+\epsilon)) - \mathcal{M}(\hat{h}_2(\tau^* - \epsilon)) < \mathcal{M}(\hat{h}_1(\tau^*+\epsilon)) - \mathcal{M}(\hat{h}_1(\tau^* - \epsilon))$.

As we’ve seen in section 2, those tasks with higher equipment intensities are consistent with routine intensive tasks, and hence the proposition says that shrinking routine employment can occur from the decrease in the price of equipment.

The condition, small enough $\alpha'_h(\tau)$, is assumed because the impacts of changing equipment price on human capital depends on the relative size of $\alpha_e$ to $\alpha_h$, not $\alpha_e$ only. The condition is only sufficient, however, not necessary. As we will show in numerical examples, $\alpha'_h(\tau)$ need not be too small, quantitatively.

Intuition under the proposition is following. A decrease in the price of equipment ($q_e$) increases the productivity of all tasks, but more for the tasks with higher equipment intensities. When the production is complementary in tasks ($\epsilon < 1$), the rise of relative productivity causes factors to flow out to other tasks, which implies the polarization.

The mechanism herein is similar to that in Lee and Shin (2017). What is different in our model is that the exogenous change generating polarization is not a shock to certain tasks only but shock affecting all tasks but in a different magnitude, where the

---

**Fig. 4.1: Equilibrium comparison: $q_{1e}$ vs $q_{2e} < q_{1e}$**

(a) Assignment

(b) Changes in employment
differences in the magnitude come from new data observation that routine intensive workers use equipment intensively. Another feature we highlight is that we have another type of task specific technological change, other than routinization, which we explore in the following propositions.

**The rise of software**

The profits from providing software and equipment are proportional to the demand of software and equipment. The demand is, in turn, proportional to the task output times factor intensity of the task. Hence, if there happened any changes in the relative size of task production, the profit from providing each type of capital would respond differently according to the corresponding factor intensity.

We already know from the proposition 1 that employment share around $\tau^*$ (in the middle) shrinks. As long as $\alpha_h'(\tau)$ is small, this means that the share of task production around $\tau^*$ has to decrease as well. Meanwhile, $\alpha_e(\tau^*) > \alpha_s(\tau^*)$, together with $\alpha_e(\bar{\tau}) < \alpha_s(\bar{\tau})$ and $\alpha_e(0) = \alpha_s(0)$ (assumption 4), imply that the decrease in the production share around $\tau^*$ actually decreases $e$ more than $s$ and the increase in the production share around $\bar{\tau}$ raises $s$ more than $e$, meaning that providing software becomes more profitable for innovators. Innovators then do more innovation toward software resulting in higher $N_s/N_e$ in the new steady state.

Although this prediction is valid for most of reasonable quantifications, we have to impose tight restrictions on the structure of intensities over the entire range on $\tau \in [0, \bar{\tau}]$ to prove analytical proposition, as we are comparing the ratio of two integrations over all $\tau$ $(\pi_s/\pi_e \propto \int \alpha_e(\tau)p(\tau)T(\tau)d\tau / \int \alpha_s(\tau)p(\tau)T(\tau)d\tau)$. To express analytical proposition in a simpler way, we consider an approximation with three discrete tasks ($j = 0, 1, 2$ for low, middle, and high) for this subsection. To be specific, consider a production technology given by

$$Y = \left( \sum_j \gamma_j^h T_j^{\epsilon_j-1} \right)^{-\epsilon_h^{-1}}$$

for $j = 0, 1, 2$, (22)

with $T_j = (b(h, j)l(h))^{\alpha_h,j} E^{\alpha_e,j} S^{\alpha_s,j}$. Detailed derivation of the equilibrium conditions with this approximation is in appendix B.

With this approximation, assumption 1 and 4 are replaced by the followings.

**Assumption 5 (distribution-II)** The measure $\mathcal{M} : [1, \bar{h}] \to [0, 1]$ has a differentiable p.d.f. $\mu(h)$ where $\mu(h)$ sufficiently small everywhere.
Assumption 6 (intensities-II)  The discrete intensities satisfy the following.

6.1 \( \frac{\alpha_{e,1}}{\alpha_{b,1}} > \frac{\alpha_{s,0}}{\alpha_{b,0}} \approx \frac{\alpha_{s,2}}{\alpha_{b,2}} \).

6.2 \( \alpha_{e,0} \approx \alpha_{s,0}, \alpha_{e,1} > \alpha_{e,2}, \) and \( \alpha_{s,2} > \alpha_{s,2} \).

In assumption 5, we add \( \mu(h) \) to be sufficiently small to think this discretization as an approximation of continuous tasks matched with a continuum of skills\(^3\).

Assumption 6.1 says that task 1 is the most equipment intensive, relative to labor, compared to task 0 and task 2. Assumption 6.2 says that the middle-skill task uses equipment more than software, the high-skill task uses software more than equipment, and the low-skill task uses software and equipment similarly.

Again, consider an exogenous decrease in the price of equipment, \( q_{1e} > q_{2e} \). Denote total varieties in the previous steady state as \( N_{s1} \) and \( N_{e1} \), and those in the new steady state as \( N_{s2} \) and \( N_{e2} \). Then we have:

**Proposition 2 (Rise of software)** Consider \( q_{1e} > q_{2e} \) with discretized tasks (22), where equipment variety were at least as many as software variety in the original equilibrium (\( N_{e1} \geq N_{s1} \)). Suppose \( \epsilon < 1 \) and assumption 2, 5, and 6. In the new steady state, software variety increases more than equipment variety, i.e., \( N_{s2}/N_{e2} > N_{s1}/N_{e1} \).

**Proof** In appendix C.

**Skill demand reversal**

Now we show that the rise of \( N_s \) leads to skill demand reversal (i.e., decrease in the demand for high skilled labor). To show that, we consider \( N_{s2} > N_{s1} \) and denote \( \hat{h}_1 \) and \( \hat{h}_2 \) be equilibrium assignment corresponding to \( N_{s1} \) and \( N_{s2} \), respectively.

**Proposition 3 (Skill demand reversal)** Consider \( N_{s2} > N_{s1} \) and suppose \( \epsilon < 1 \) and assumption 1 to 4. With small enough \( \alpha'_{h} (\tau) \), the matching function shifts upward everywhere, i.e., \( \hat{h}_2 (\tau) > \hat{h}_1 (\tau) \) for all \( \tau \in (0, \bar{\tau}) \).

**Proof** In appendix C.

---

\(^3\)With the discretized tasks, the assignment function \( \hat{h}(\tau) \) becomes a sequence of threshold human capital \( \hat{h}_j \). When \( \hat{h}_j \)'s changes, it not only affects demand for labor around the threshold level but also affects total labor supply given to each task, \( \int_{\hat{h}_{j-1}}^{\hat{h}_j} \mu(h)dh \). We ignore indirect effects coming from changes in \( \int_{\hat{h}_{j-1}}^{\hat{h}_j} \mu(h)dh \) by assuming \( \mu(\hat{h}_j) \) and \( \mu(\hat{h}_{j-1}) \) are small enough.
Fig. 4.2: Equilibrium comparison: $N_{s1}$ and $N_{s2} > N_{s1}$

(a) Assignment

(b) Changes in employment

Note that an increase in the size of variety raises the productivity of software intensive tasks relatively more than others (equation (11)). Following the same intuition with the case of polarization, this would lead to a reallocation of labor out of the high-skill task to lower skilled tasks under complementarity ($\epsilon < 1$). The change in assignment function is depicted in figure 4.2, showing all workers downgrade their task.

This proposition, together with proposition 2, imply that there would be skill demand reversal following from the rise of software on top of ongoing polarization when the price of equipment falls. Note that proposition 2 shows that relative size of software variety to equipment variety ($N_s/N_e$) would increase. One needs to be cautious since this does not mean that only $N_s$ is increasing. In fact, when $q_e$ falls $N_e$ also should rise as otherwise $\pi_e/\eta_e > \rho$. The proposition says that the variety of software would rise more than that of equipment. Since the rise of $N_e$ will lead to polarization, the resulting steady state equilibrium itself would be the mix of polarization and skill demand reversal.

Again, the technical assumption, small enough $\alpha'_h(\tau)$, is required for proof, but this needs not be that small quantitatively. To see the comparative statics with illustrations, we bring some numerical examples in appendix D.

**CES task production**

To characterize analytical results, we assumed unitary elasticity between labor and capital. However, what is crucial is the elasticity of substitution between labor and
Fig. 4.3: Equilibrium comparison with $q_e = 1$ and $q_e = .2$: CES task production

(a) Changes in employment with $\sigma_e = 1.2$

(b) Changes in employment with $\sigma_s = 1.2$

capital greater than the elasticity of substitution between tasks ($\epsilon$), and we expect that the task production needs not be Cobb-Douglas in generating responses consistent with the proposition 1 to 3, at least numerically.

What could we expect if the elasticity of substitution between labor and capital is different from one? We predict the larger the elasticity of substitution between labor and equipment becomes, the higher polarization effect would appear. We also expect the skill demand reversal effect (decreasing high-skill demand), as well as the rise of software, would be enhanced when the elasticity of substitution between labor and software is bigger.

Intuitively, when the elasticity of substitution between equipment and labor is greater than one, a decrease in the price of equipment lowers demand for the middle-skill tasks not only through the adjustment in the assignment but also through the adjustment between labor and equipment within a task. Also, when the elasticity of substitution between software and labor is greater than one, corresponding increases in the software would substitute high-skill labor more than before.

To confirm the intuition, we provide several numerical illustrations in figure 4.3 (details are in appendix D). As expected, the magnitude of shrinking middle becomes bigger with increases in $\sigma_e$ and decreases in high skill demand is enhanced with increases in $\sigma_s$. 
5 Empirical Evidence

This section checks a validity of the model’s prediction using industry data. To be specific, we test two predictions. First, the model predicts a negative relationship between changes in the relative price of software to equipment and changes in middle-skill employment relative to high-skill employment. Second, the model implies a positive correlation between changes in the relative price of software to equipment and changes in software innovation relative to other innovation.

We measure the relative price of equipment to software from dividing nominal investment by real investment, provided by BEA. For the relative employment of middle-skill to high-skill occupations, we use the employment of routine occupations divided by the employment of cognitive occupations by industry, computed from Census data. Finally, the relative size of software innovation to other innovation is measured by own account software investment (in-house software investment by firms) divided by R&D excluding software. Detailed explanation on data construction is in appendix E.

Figure 5.1 shows differences in the growth of middle-skill and high-skill employment against changes in the software price relative to equipment. Figure 5.2 is changes in software innovation net of R&D expenditures excluding software against changes in the relative price. The first has a negative and the second has a positive relation, consistent with the model’s predictions.

To see if these relations are statistically significant, we estimate following regression:

$$\Delta \log y_{i,t} = a + c_t + \Delta \log (q_{s,i,t}/q_{e,i,t}) + \varepsilon_{i,t},$$

where $y_{i,t}$ is either ratio of routine (middle-skill) employment to cognitive (high-skill) employment or ratio of in-house software investment to R&D expenditures excluding software. The estimation results are in table 5.1, showing significant relations between the two variables.

<table>
<thead>
<tr>
<th>Tab. 5.1: Estimation results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Routine/Cognitive</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>Sft price / Eqp price</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Fixed Effects</td>
</tr>
<tr>
<td>$R^2$</td>
</tr>
</tbody>
</table>

$p$-values in parentheses.
Fig. 5.1: Changes in the relative price and the relative employment

Fig. 5.2: Changes in the relative price and the relative innovation
6 Quantitative Analysis

In this section, we use the discretized model (appendix B) to map the tasks in the model to ten occupational groups consistent with one digit SOC code. Specifically, following production technology is used for the quantitative analysis.

\[ Y = \left( \sum_{j} \gamma_j T_j \right)^{1/\alpha_s} \]

where \( T_j = A_j \left[ \left( \alpha_{h,j} \int_{h_{j-1}}^{h_j} b(h, j) \mu(h) dh \right)^{\sigma_e-1} \sigma_e + \alpha_{e,j} E_j^{\sigma_e-1} \right] + \alpha_{s,j} S_j^{\sigma_s-1} \)

We do mainly three exercises in this section. The first is to investigate how much the channel of endogenous innovation of software can explain changes in the pattern of polarization. For this exercise, we exactly match the changes in the employment share of middle-skill occupations and compare the employment dynamics of high- and low-skill occupations generated from the model with innovation and without innovation. To match the middle-skill employment, we use changes in task specific technical change \( A_j \) for middle-skill occupations in addition to the price of equipment and software (\( q_e \) and \( q_s \)).

For the benchmark exercise above, we calibrate constant \( \nu_e \) and \( \nu_s \). However, changes in the market structure can also be an important factor affecting innovation incentives. For this reason, we also ask how much time-varying markup’s observed in data additionally affect the main result above as second exercise.

The last question is how much of changes in the decline of equipment price only could account for the shifts in the employment share between occupations. For this, we do simulation again with all other parameters fixed, assuming constant \( A_j \) for all middle-skill occupations with changes in \( q_e \) only.

6.1 Calibration

We calibrate most of the parameters to 1980 data moments assuming steady state. For the functional forms, we set productivity function \( b(h, j) \) as

\[ b(h, j) = \begin{cases} \hat{h} & \text{if } j = 0, \\ h - \chi_j & \text{if } j \geq 1. \end{cases} \]
and the skill distribution $M(h)$ as

$$M(h) = 1 - h^{-a}.$$ 

Weight parameters in the final production ($\gamma_j$'s) are from employment share by occupation in 1980. The $\chi_j$'s and $a$ are determined to match income share across occupational groups in 1980. Between factor intensities by tasks ($\alpha_h$, $\alpha_e$, $\alpha_s$) are matched to equipment and software investment by occupational group\textsuperscript{4}, and labor share in 1980. For the benchmark analysis, we map equipment in the model to industrial equipment in data, as it has the closest relation with the routine-ness of occupations (figure 2.3).

There are two categories of parameters that are difficult to identify only from 1980 moments: (1) the elasticity of substitution ($\epsilon$, $\sigma_e$ and $\sigma_s$), and (2) markup related parameters ($\nu_e$ and $\nu_s$). We use various methods to identify these parameters.

For the elasticity of substitution between factors in task production ($\sigma_e$ and $\sigma_s$), I match the level of labor share in 2010, with and without software capital. To see how identification works, note that factor share in a given task $\tau$ can be derived as following:

$$LS_s = \frac{wL}{wL + p_e \tilde{E}} = \frac{1}{1 + \left(\frac{\alpha_e}{\alpha_h}\right)^{\sigma_e} \left(\frac{q_e}{\nu_e N_e^e \omega}\right)^{1-\sigma_e}},$$

$$LS = \frac{wL}{wL + p_e \tilde{E} + p_s \tilde{S}} = \frac{1}{1 + \left(\frac{\alpha_e}{\alpha_h}\right)^{\sigma_e} \left(\frac{q_e}{\nu_e N_e^e \omega}\right)^{1-\sigma_e} + \left(\frac{q_s}{\nu_s N_s^s \omega}\right)^{1-\sigma_s}}.$$  

where $LS_s$ is labor share without software and $LS$ is standard labor share. From equation (24), it is straightforward to see that labor share without software does not directly depend on the elasticity of substitution between labor and software, $\sigma_s$.

What makes this strategy even more useful is the fact that labor share with software and without software show different trends from 1980 to 2010. The labor share without software has no clear trend in it while total labor share has a declining trend (Koh et al., 2016). Given that labor share without software capital has no clear trend, it is easy to

\textsuperscript{4}We assume that the number of commodities used by each occupation same and attribute capital investment in 1980 to each occupation.
predict that the elasticity of substitution between labor and equipment would be close to 1 ($\sigma_e \approx 1$). Moreover, we could expect $\sigma_s > 1$ from the decreasing trend of total labor share and increasing trend of software innovation.

For the mark up related parameters $\nu_e$ and $\nu_s$, we estimate them separately using Industry Account and Fixed Asset Table from BEA, following Domowitz et al. (1988). To be specific, we estimate

$$\Delta \log q_{it} - \alpha_L \Delta \log l_{it} - \alpha_m \Delta \log m_{it} = c_i + b \Delta \log q_{it} + \varepsilon_{it},$$

where $q$ is gross output/capital, $l$ is employment/capital, $m$ is intermediate input/capital, $\alpha_L$ and $\alpha_M$ is labor and intermediate share, respectively. We estimate this relation for equipment producing industries (industry 3 in BEA industry codes) and software producing industry (industry 511). To control for endogeneity, GDP growth is used as an instrumental variable. Once estimated, $\nu_e$ and $\nu_s$ can be obtained by calculating $1 - b$. The estimation results are in table 6.1.

The most difficult one to identify is the elasticity of substitution between tasks, $\epsilon$. For this parameter, we use 0.7 from Lee and Shin (2017), and do robustness check with varying the value of $\epsilon$ in subsection 6.3. When we target trends in between occupational moment, such as income share, we would estimate $\epsilon$ lower than 0.7. As will be shown later, the lower $\epsilon$ is, the more the model explains data. In this sense, the benchmark calibration is conservative.

Table 6.2 and 6.3 summarizes all the calibration results.

### 6.2 Simulation results

#### 6.2.1 The pattern of job polarization

We assume the economy was in the steady state in 1980, and compute new steady state corresponding to the changes in the price of equipment. For the main exercise,
we also add exogenous productivity changes in the middle (i.e., routine-biased technical change not in the price of capital), matching changes in the employment share of routine occupations. We then ask: how much the model explain shifts in the trend of high skilled and low skilled employment with and without endogenous software innovation?

Figure 6.1 shows the simulation results. Figure 6.1a displays annualized changes in employment during first two decades (blue bar) and that in last decade (light blue bar) by occupational group, observed in data. Figure 6.1b and 6.1c show the same series generated from the model with endogenous innovation (varying $N_e$ and $N_s$) and without innovation (no changes in $N_e$ and $N_s$), respectively.

The blue bar (changes during 1980-2000) is higher than the light blue bar (changes during 2000-2010) for cognitive occupations and lower for low-skilled services occupations, as we have highlighted in section 2. As can be seen in Figure 6.1b and 6.1c, these changes in the pattern have appeared only in the simulation with endogenous innovation, i.e., with increases in $N_s/N_e$. In data, the increase of cognitive (high-skill)

<table>
<thead>
<tr>
<th>Occupational Group</th>
<th>$\alpha_e$</th>
<th>$\alpha_s$</th>
<th>$\alpha_h$</th>
<th>$\gamma$</th>
<th>$\chi$</th>
</tr>
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<tbody>
<tr>
<td>Low skilled services</td>
<td>0.12</td>
<td>0.06</td>
<td>0.82</td>
<td>0.09</td>
<td>0.00</td>
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<td>Sales</td>
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</tr>
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<td>0.03</td>
<td>0.07</td>
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<tr>
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<td>0.05</td>
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<td>0.76</td>
<td>0.09</td>
<td>0.10</td>
</tr>
<tr>
<td>Management</td>
<td>0.03</td>
<td>0.26</td>
<td>0.71</td>
<td>0.07</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Target

| Equipment, software, and labor share | Employment Income share |

Tab. 6.3: Remaining parameters

<table>
<thead>
<tr>
<th>Value</th>
<th>Obtained from</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_s$</td>
<td>1.19</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>1.00</td>
</tr>
<tr>
<td>$\nu_e$</td>
<td>0.77</td>
</tr>
<tr>
<td>$\nu_s$</td>
<td>0.53</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.70</td>
</tr>
</tbody>
</table>
occupation during the last decade has been 0.30%p lower than the average of first two decades. In the model, it has become 0.12%p lower with endogenous innovation and only 0.03%p less without endogenous innovation. For low-skilled services occupation, the change in the increases between 2000-2010 and 1980-2000 was +0.26%p in data where that in the full model appeared to be +0.07%p. The model without innovation even produces a negative change of −0.01%p (table 6.4).

Another way to look at the simulation results are the employment dynamics by occupation from 1980 to 2010, represented in figure 6.2. The deviation from trend in high-skill occupation in the model captures 56% of actual deviation in data (figure 6.2a and 6.2b), where deviation from trend in low-skill employment in the model is 39% of data (figure 6.2c and 6.3a), with endogenous innovation. Not only the magnitude, but the model also captures relatively well for the timing of changes in the trends, as it produces much bigger changes during 2000-2010 than first two decades, again, consistent with data. Without endogenous innovation, the simulation generates almost no variations in the trends of high- and low-skill employment.

### 6.2.2 The rise of software

In data, the ratio between software investment to industrial equipment investment increases from 0.16 to 1.7, more than ten times higher. Since we match the initial level of the relative investment 0.16 exactly by calibration, we compare the level of the

![Fig. 6.1: Simulation results – changes in the employment shares](image)
### Tab. 6.4: Simulation results

Changes in the shares of employment (%p, annualized)

<table>
<thead>
<tr>
<th>Data Model</th>
<th>1980-1990 (A)</th>
<th>1990-2000 (B)</th>
<th>2000-2010 (C)</th>
<th>((C-(A+B)/2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cognitive</td>
<td>0.43</td>
<td>0.36</td>
<td>0.09</td>
<td>-0.31</td>
</tr>
<tr>
<td></td>
<td>0.22</td>
<td>0.15</td>
<td>0.06</td>
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<td></td>
<td>0.27</td>
<td>0.25</td>
<td>0.22</td>
<td>-0.04</td>
</tr>
<tr>
<td>Low skilled services</td>
<td>0.04</td>
<td>0.10</td>
<td>0.34</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>0.27</td>
<td>0.31</td>
<td>0.36</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>0.22</td>
<td>0.22</td>
<td>0.21</td>
<td>-0.01</td>
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</tbody>
</table>

Deviations from linear trend between 1980 to 1990 (%, %p)

<table>
<thead>
<tr>
<th>Data Model</th>
<th>Cognitive Share in 2010 (A)</th>
<th>Trend implied (B)</th>
<th>(A-B)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>33.8</td>
<td>38.0</td>
<td>-4.16</td>
</tr>
<tr>
<td></td>
<td>29.3</td>
<td>31.6</td>
<td>-2.30</td>
</tr>
<tr>
<td></td>
<td>32.4</td>
<td>33.0</td>
<td>-0.64</td>
</tr>
<tr>
<td>Low skilled services</td>
<td>Share in 2010 (A)</td>
<td>18.0</td>
<td>14.5</td>
</tr>
<tr>
<td></td>
<td>22.6</td>
<td>21.3</td>
<td>1.33</td>
</tr>
<tr>
<td></td>
<td>19.8</td>
<td>19.9</td>
<td>-0.09</td>
</tr>
</tbody>
</table>
ratio in 2010 to check how well the model does in explaining the rise of software. The full model with innovation explains 56% of increases in the rise of software investment relative to the equipment (figure 6.3b). If we shut down the endogenous innovation channel (i.e., no changes in $N_s$ and $N_e$), the model generates less than half of software to equipment ratio compared to the model with innovation (green dashed line).

### 6.2.3 Time-varying markups

So far, we’ve assumed constant $\nu_e$ and $\nu_s$. However, price to cost margins of equipment and software producing industries exhibit different trends (figure 6.4). These changes in market structure also affect innovational incentives of innovators. Specifically, as software industry shows faster increase in the price to cost margin compared to equipment producing industries, innovation toward software would become more profitable.
To examine how important the changes quantitatively, we map the variations in the trend of price to cost margin into changes in $\nu_e$ and $\nu_s$. The simulation result is in table 6.5.

Again, the deviation from trend of cognitive employment share in the model is 56% of data with fixed markup parameters. The deviation from trend goes up to 60% of data with time-varying markups. Without innovation, the model generates only 15% of data with fixed $\nu$’s and 9% of data with time-varying $\nu$’s, meaning that additional explanation of innovation channel becomes 9%p bigger with time-varying markups. The variations in the markups do not add much explanatory power on the deviation from trend of low skilled services employment share and software to equipment investment ratio.

<table>
<thead>
<tr>
<th>Tab. 6.5: Ratio of model variables to data</th>
<th>Deviation from trend</th>
<th>Software/Equipment</th>
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<tr>
<td></td>
<td>Cognitive</td>
<td>Low services</td>
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<tr>
<td>Fixed $\nu$</td>
<td>Innov (A)</td>
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<td>No innov (B)</td>
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<td></td>
<td>A-B</td>
<td>0.41</td>
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<tr>
<td>Time-varying $\nu$</td>
<td>Innov (A)</td>
<td>0.60</td>
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<tr>
<td></td>
<td>No innov (B)</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>A-B</td>
<td>0.50</td>
</tr>
</tbody>
</table>
Fig. 6.4: Price to cost margin\(^1\) of equipment and software producing industries\(^2\)

![Graph showing price to cost margin over time for equipment and software producing industries.]

**Notes:** 1) (gross output-intermediates-compensation of employees)/gross output. 2) The equipment producing industries are 331, 332, 333, 334, 335, 3361MV, 3364TO, 337, and 339. The software producing industry is 511. 3) Solid line is HP-trend with \(\lambda = 100\).

### 6.2.4 The decline in the price of equipment only

Recall that we use exogenous variation in middle-skill specific technical change (changes in \(A_j\)) in addition to the decrease in the price of capital observed in data to match changes in the employment share of middle-skill occupations exactly. The second exercise is excluding all the exogenous productivity changes in the middle, and then see how much of the decline in the price of equipment only explains variations in the share of employment by occupation.

The result is in figure 6.5. The changes in equipment price explain 6%, 21%, and 10% of cognitive, routine-manual (i.e., transportation, mechanics, machine operators, and construction and production), and low skilled services employment. Two things are noteworthy.

First, the directions of changes are consistent with data, except for the administrative occupations. Note that our data construction assumes that the amount of capital item used by a certain occupation being proportional to the number of specific items used by the occupation in the investment category. We may underestimate the impact of the capital embodied technical change on a certain occupation if that occupation uses a small number of items most of the time, but that item has been affected by the technical change a lot, for example, book-keepers that are relying mostly on a single
item PC. We think that this might be a case for the administrative workers.

Second, despite the consistency in directions, the observed decline of equipment price explains only one-tenth to one-fifth of data concerning magnitude. One possible reason is a potential underestimation of the technical change embodied in the price of industrial equipment, which we use as a benchmark equipment series. Note that the quality-adjustment has been done following the method in Cummins and Violante (2002), which econometrically extends Gordon (1990)'s quality-bias, covering 1957 to 1984. One problem is that the start of routine-biased technical change is usually considered as the early 1980s, which is rarely covered by Gordon (1990)'s sample periods.

What would the number become if we use alternative price series? When we use the changes in the price of information-processing equipment\footnote{Information-processing equipment includes computer and peripheral equipment including PC of which price is measured with quality-adjustment by BEA.} instead of that of industrial equipment, for example, the magnitude goes up to 18%, 79%, and 38% of cognitive, routine-manual, and low skilled services employment, respectively. This is almost four times bigger than the previous case.

The analysis implies two further studies might be helpful in understanding changes in the occupational structure generated by capital-embodied technical change. The first is to make a meaningful distinction between capital types within equipment category. The second is an attempt to measure a better quality-adjusted price for various capital items.
6.2.5 The decline of labor share

Although labor share dynamics is not a primary of this exercise, it merits a further discussion. Note that we are using changes in labor share as a target variable for calibration of the elasticity of substitution ($\sigma_e$ and $\sigma_s$), and hence it’s not surprising that the labor share in the model exactly matches labor share in data in both 1980 and 2010. What is new is that the simulation without endogenous software innovation produces almost flat labor share (figure 6.3b).

This happens because our identification strategy gives the elasticity of substitution between equipment and labor ($\sigma_e$) close to one, and exogenous variation does not generate declining labor share without software innovation. It means that the declining labor share in our model is mostly coming from the software investment. We would like to highlight that we find the negative correlation between software investment and the labor share not only in time series (figure 6.6a) but also in industrial variations, especially during 2000-2010 (figure 6.6b and 6.6c). We believe that the detailed investigation of the relation between labor share and software capital can be a meaningful future research.

6.3 Sensitivity

In this subsection, we check how the results vary by the elasticity of substitution between tasks ($\epsilon$) and markup-related parameters ($\nu_e$ and $\nu_s$). Again, we can not identify the elasticity of substitution between tasks ($\epsilon$) from cross-sectional information.

The most straightforward data component related to $\epsilon$ is changes in employment shares by occupation. However, the variable is our main interests and hence it is not available to use them directly. Therefore, we use $\epsilon$ borrowed from other paper, which is 0.7. Since the model mechanism is amplified when the tasks are more complement, we would expect that the model’s explanatory power becomes larger as $\epsilon$ gets smaller. Table 6.6 confirms this intuition.

Another group of parameters that we can consider an alternative is the markup-related parameters, $\nu_e$ and $\nu_s$. In the estimation, software related markup was less significant than equipment related markup, maybe due to a smaller sample size (table 6.1). As an alternative, we measure the price-cost margin by \((value added - compensation of employees) / (value added + intermediate input)\). With this alternative, $\nu_s$ becomes a bit higher than before while $\nu_e$ does not change much. Table 6.6 shows that the model explains less of the changes in the pattern of job polarization while it explains a bit more of the rise of software with this alternatives.
Fig. 6.6: Labor share and software

(a) Labor share trend

(b) By industry – Manufacturing

(c) By industry – Services

Note: 1) The labor share without software capital is constructed following the method in Koh et al. (2016).
2) Industry 514 (with changes in labor share larger than 1 in both periods) has been excluded in this figure.

7 Conclusion

We provided a model with the heterogeneous task and two types of capital of which varieties are determined endogenously through firm’s innovation. We showed both analytically and quantitatively that the mechanism in the model is important in understanding the impacts of capital-augmented technical change on the structure of labor market.

One important implication is that two types of capital – software and equipment—measured in National Accounts is a good proxy for the recent technological changes. Understanding the impact of a technical change to the economy has always been an important topic. One of the main difficulties is that technological change is not easy to measure especially regarding aggregate analysis. This paper shows that investigation of
Tab. 6.6: Robustness

The elasticity of substitution between tasks ($\epsilon$)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Cognitive benchmark</td>
<td>-4.16</td>
<td>-2.30 (.55)</td>
<td>-0.64 (.15)</td>
<td>-0.03 (.01)</td>
</tr>
<tr>
<td>(dev. from trend) $\epsilon = .5$</td>
<td>-4.16</td>
<td>-2.47 (.59)</td>
<td>-0.64 (.16)</td>
<td>-0.06 (.02)</td>
</tr>
<tr>
<td>$\epsilon = .3$</td>
<td>-4.16</td>
<td>-2.66 (.64)</td>
<td>-0.67 (.16)</td>
<td>-0.11 (.03)</td>
</tr>
<tr>
<td>$\epsilon = .1$</td>
<td>-4.16</td>
<td>-2.81 (.68)</td>
<td>-0.69 (.17)</td>
<td>-0.16 (.04)</td>
</tr>
</tbody>
</table>

| Low skilled benchmark | 3.50 | 1.33 (.38) | -0.09 (-.03) | -0.01 (-.00) | -0.01 (-.00) |
| (dev. from trend) $\epsilon = .5$ | 3.50 | 1.56 (.45) | -0.08 (-.02) | -0.03 (-.01) | -0.03 (-.01) |
| $\epsilon = .3$ | 3.50 | 1.82 (.52) | -0.08 (-.02) | -0.05 (-.01) | -0.06 (-.02) |
| $\epsilon = .1$ | 3.50 | 2.06 (.59) | -0.07 (-.02) | -0.08 (-.02) | -0.09 (-.03) |

| Soft/eqp benchmark | 1.68 | 0.94 (.56) | 0.42 (.25) | 0.20 (.12) | 0.18 (.11) |
| (lev. in 2010) alternative $\epsilon = .5$ | 1.68 | 0.98 (.59) | 0.43 (.26) | 0.22 (.13) | 0.19 (.11) |
| $\epsilon = .3$ | 1.68 | 1.02 (.61) | 0.44 (.26) | 0.23 (.14) | 0.20 (.12) |
| $\epsilon = .1$ | 1.68 | 1.05 (.62) | 0.45 (.27) | 0.24 (.15) | 0.21 (.13) |

Markup related parameters ($\nu_e$ and $\nu_s$)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Cognitive benchmark</td>
<td>-4.16</td>
<td>-2.30 (.55)</td>
<td>-0.64 (.15)</td>
<td>-0.03 (.01)</td>
</tr>
<tr>
<td>(dev. from trend) alternative</td>
<td>-4.16</td>
<td>-1.72 (.41)</td>
<td>-0.85 (.21)</td>
<td>-0.02 (.00)</td>
</tr>
<tr>
<td>Low skilled benchmark</td>
<td>3.50</td>
<td>1.33 (.38)</td>
<td>-0.09 (-.03)</td>
<td>-0.01 (-.00)</td>
</tr>
<tr>
<td>(dev. from trend) alternative</td>
<td>3.50</td>
<td>0.77 (.22)</td>
<td>0.10 (.03)</td>
<td>-0.01 (-.00)</td>
</tr>
<tr>
<td>Soft/eqp benchmark</td>
<td>1.68</td>
<td>0.94 (.56)</td>
<td>0.42 (.25)</td>
<td>0.20 (.12)</td>
</tr>
<tr>
<td>(lev. in 2010) alternative</td>
<td>1.68</td>
<td>0.96 (.57)</td>
<td>0.59 (.35)</td>
<td>0.19 (.11)</td>
</tr>
</tbody>
</table>

Alternative $q_e$'s

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Cognitive benchmark</td>
<td>-4.16</td>
<td>-2.30 (.55)</td>
<td>-0.64 (.15)</td>
<td>-0.03 (.01)</td>
</tr>
<tr>
<td>(dev. from trend) Total eqp</td>
<td>-4.16</td>
<td>-1.94 (.47)</td>
<td>-0.40 (.10)</td>
<td>-0.14 (.03)</td>
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<tr>
<td>Ind+IT</td>
<td>-4.16</td>
<td>-2.12 (.51)</td>
<td>-0.88 (.21)</td>
<td>-0.32 (.08)</td>
</tr>
<tr>
<td>Low skilled benchmark</td>
<td>3.50</td>
<td>1.33 (.38)</td>
<td>-0.09 (-.03)</td>
<td>-0.01 (-.00)</td>
</tr>
<tr>
<td>(dev. from trend) Total eqp</td>
<td>3.50</td>
<td>1.37 (.39)</td>
<td>-0.03 (-.01)</td>
<td>-0.05 (-.02)</td>
</tr>
<tr>
<td>Ind+IT</td>
<td>3.50</td>
<td>2.04 (.58)</td>
<td>0.52 (.15)</td>
<td>0.29 (.08)</td>
</tr>
<tr>
<td>Soft/eqp benchmark</td>
<td>1.68</td>
<td>0.94 (.56)</td>
<td>0.42 (.25)</td>
<td>0.20 (.12)</td>
</tr>
<tr>
<td>(lev. in 2010) Total eqp</td>
<td>1.68</td>
<td>0.98 (.59)</td>
<td>0.43 (.26)</td>
<td>0.22 (.13)</td>
</tr>
<tr>
<td>Ind+IT</td>
<td>1.68</td>
<td>1.02 (.61)</td>
<td>0.44 (.26)</td>
<td>0.23 (.14)</td>
</tr>
</tbody>
</table>
different types of capital can be a meaningful effort to capture the recent technological changes.

Another important implication of this paper is that a technological change affecting a small group of occupations would lead to other types of innovation eventually affecting a broader set of occupations. Note that the same intuition would apply to a sectoral technical change. This paper analyzes the technical change in the context of task-biased technological change, but a task biased technical change has a tight link to a sector-biased technical change as different sectors use a different combination of occupations, emphasized in Lee and Shin (2017) or Bárány and Siegel (forthcoming).

Our model would have many useful extensions which could be implemented easily. For example, further decomposition of equipment capital into sub categories will be helpful in understanding more detailed changes in occupational structure. As our primary source of exogenous variation (changes in the price of equipment) is neither task-specific nor sector-specific, integrating multi-sector structure would give other interesting implications in the relation between polarization and structural change or evolution of task-specific and sector-specific productivity.

Though not as straightforward, the analysis herein would lead to many interesting future research topics as well. For example, using firm-level software and equipment investment data, we may generate interesting implications on the impacts of technological change on firm level heterogeneity together with occupation level heterogeneity. As more and more countries are trying to broaden types of capital measured in National Accounts (System of National Account 2008), a multi-country extension could also be available, enabling the analysis of trade or offshoring in addition to the technological changes.
References


Appendices

A Use of equipment and software by occupation

The capital use by occupation data is constructed by combining BEA NIPA and O*NET Tools and Technology Database. In NIPA table 5.5, the investment on non-residential equipment are categorized by 25 types. In UNSPSC, the classification system used in O*NET Tools and Technology database, there are 4,300 commodities, which are in 825 classes, in 173 families, and in 36 segments.

To construct a mapping between two, we firstly assign one of NIPA investment types to the relevant segment in UNSPSC. Often, it is apparent that a segment includes several types of equipment investment in NIPA. In this case, we use the family categories in the assignment procedure. Again, if a family apparently includes several types in NIPA, we use classes. Through this procedure, we could make a rough concordance between a subset of UNSPSC and the types of equipment investment in NIPA. The constructed concordance is shown in table A1.

Next, we assume that two tools have same price if they are classified in the same category. For example, the “metal cutting machines” in UNSPSC is assigned to “metalworking machinery” in NIPA investment type. The value of using the metal cutting machines are then the amount of investment in metalworking machinery divided by total use of all the commodities in the metalworking machinery category, where the total use of all the tools in the metalworking machinery is defined as sum of a number of total employment of each occupation times a number of UNSPSC commodities assigned to the metalworking machinery that each occupation uses.

The method is assuming that the number of tools above well represent the value of them, only within the NIPA investment category. Across the NIPA investment categories, each number of tools used would get different weights, according to the average amount of investment given to each tool. The procedure may make a big difference from average number of tools if a category with many commodities had small values compared to a category with few commodities. However, as more differentiated categories are usually advanced (and hence have expensive items), we expect not much difference from the adjustment.
<table>
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<tr>
<th>Line</th>
<th>NIPA Line Title</th>
<th>Code</th>
<th>UNSPSC Code</th>
<th>UNSPSC Title</th>
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<td>43210000</td>
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<td>Computer Equipment and Accessories</td>
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<td>43190000,</td>
<td>Communications Devices and Accessories, Audio and visual presentation and</td>
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<td>45110000</td>
<td>composing equipment</td>
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<td>Printing and publishing equipment, Photographic or filming or video</td>
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<td>equipment</td>
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<td>+ General industrial, including materials handling, equipment</td>
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<td></td>
<td>23190000, 23200000,</td>
<td></td>
<td>supplies, Industrial food and beverage equipment, Mixers and their parts</td>
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<tr>
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<td>23210000, 23220000,</td>
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<td>and accessories, Mass transfer equipment, Electronic manufacturing machinery and equipment and accessories, Chicken processing machinery and equipment, Sawmilling and lumber processing machinery and equipment, Industrial machine tools, Material handling machinery and equipment, Containers and storage, Moldings, Distribution and Conditioning Systems and Equipment and Components</td>
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<td>20</td>
<td>Electrical transmission, distribution, and industrial apparatus + Electrical equipment, n.e.c.</td>
<td>26101100, 26101200, 26101300, 26110000, 26120000, 26130000, 26140000, 39000000</td>
<td></td>
<td>Electric alternating current AC motors, Electric direct current DC motors, Non electric motors, Batteries and generators and kinetic power transmission, Electrical wire and cable and harness, Power generation, Atomic and nuclear energy machinery and equipment, Electrical Systems and Lighting and Components and Accessories and Supplies</td>
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<td>Motor vehicles</td>
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<td>25130000</td>
<td>Aircraft</td>
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<td>Ships and boats</td>
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<td>25110000</td>
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<td>Railroad equipment</td>
<td></td>
<td>25120000</td>
<td>Railway and tramway machinery and equipment</td>
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<td>Other equipment</td>
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<td>Furniture and Furnishings</td>
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<td>Agricultural machinery</td>
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<td>Building and Construction Machinery and Accessories</td>
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<td>Service industry machinery</td>
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B Discrete approximation of the model

This section discusses equilibrium conditions with discrete approximation of the model. For the approximation, assumption 1 and 4 are replaced by assumption 5 and 6 in section 3 and 4.

The task production is given by equation (22) with tasks discretized into \( j = 0, 1, \cdots, J \). Now the tasks are discrete, so workers are sorted into each task according to cutoff level of human capital \( \hat{h}_j \). More precisely, we have a sequence of human capital \( \{ \hat{h}_j \}_{j=0}^{J+1} \) such that a worker with \( h \in [\hat{h}_j, \hat{h}_{j+1}) \) are sorted into task \( j \) with \( \hat{h}_0 = h \) and \( \hat{h}_{J+1} = \bar{h} \).

For a worker with exactly the threshold level of human capital should be indifferent between tasks so that

\[
\omega_j b(\hat{h}_j, j) = \omega_{j-1} b(\hat{h}_j, j - 1), \quad \text{for all } j, \quad \text{for } j = 1, \cdots, J \tag{B.1}
\]

replacing the original equilibrium condition (9).

The task production is solving

\[
\max p_j T_j - \int_{\hat{h}} w(h) l(h) dh - \int_{k=0}^{N_e} p_e(k) e(k) dk - \int_{k=0}^{N_s} p_s(k) s(k) dk,
\]

which gives the FOCs,

\[
w(h) = \omega_j b(h, j) = p_j T_j^{1/\sigma_e} H_j^{1/\sigma_e} \left( \int_{\hat{h}_j}^{\hat{h}_{j+1}} \frac{1}{\sigma_e} b(h, j) \mu(h) dh \right)^{-1/\sigma_e} b(h, j),
\]

\[
q_e = p_j T_j^{1/\sigma_e} H_j^{1/\sigma_e} \left( \frac{1}{\sigma_e} (N_e) \sigma_{e, e}^{-1} - 1 \right) e_j^{-1/\sigma_e},
\]

\[
q_s = p_j T_j^{1/\sigma_e} (N_s) \sigma_s^{-1} - 1 s_j^{-1/\sigma_s},
\]

using the fact that \( p_e = q_e/\nu_e, \ p_s = q_s/\nu_s, \ e_j(k) = e_j, \text{ and } s_j(k) = s_j \) in equilibrium, and

\[
H_j := \left[ \alpha_{h,j} \left( \int_{\hat{h}_j}^{\hat{h}_{j+1}} b(h, j) \mu(h) dh \right)^{-1/\sigma_e} + \alpha_{e,j} \left( \int_{k=0}^{N_e} e(k) \nu_e dk \right)^{-1/\sigma_e} \right].
\]

Combining the FOCs, we get

\[
p_j = \left[ \left( \frac{\sigma_e}{\sigma_e - 1} \right)^{1/\sigma_e} + \alpha_{h,j} \left( \frac{q_e}{\nu_e N_e^{1/\sigma_e}} \right)^{1/\sigma_e} + \alpha_{e,j} \left( \frac{q_s}{\nu_s N_s^{1/\sigma_e}} \right)^{1/\sigma_e} \right]^{1/\sigma_e}, \quad \text{for } j = 0, \cdots, J \tag{B.2}
\]

which replaces equation (11).
The demand for each task is from
\[ \max Y - \sum_j p_j T_j, \]
which gives
\[ p_j = \left( \frac{\gamma_j Y}{T_j} \right)^{\frac{1}{\epsilon}}. \]
Combining this with FOCs, we obtain
\[ p^s_{j-\sigma_s} = \frac{\gamma_j \alpha_{h,j}^s \left( \alpha_{h,j}^{e_s} \omega_j^{1-\sigma_e} + \alpha_{e,j}^{e_s} \left( \frac{q_e}{\nu_e N_e} \right)^{1-\sigma_e} \right)^{\frac{\sigma_e}{\sigma_s - \sigma_e}} Y}{\omega_j^{\sigma_e} \int_{h_{j-1}}^{h_j} b(h, j) \mu(h) dh}, \]
for \( j = 0, \cdots, J \), (B.3)
replacing equation (10).

Now the equilibrium thresholds \( \hat{h}_j \)'s, wage rate \( \omega_j \)'s and prices \( p_j \)'s are obtained by solving equation (B.1) to (B.3), which are \( 3J + 1 \) equations with the same number of unknowns.
C Proof

**Proof of lemma 2** Since assignment function \( \hat{h}(\tau) \) is strictly increasing, its inverse \( \hat{\tau}(h) \) is well-defined. From the demand for task, equation (7), we know that there will be strictly positive task output \( T(\tau) > 0 \) (and hence \( l(h, \hat{\tau}(h)) > 0 \)) for all \( \tau \in [0, \bar{\tau}] \). The equation (8) and lemma 1 then implies

\[
w(h) = \omega(\hat{\tau}(h))b(h, \hat{\tau}(h)) \geq \omega(\hat{\tau}(h'))b(h, \hat{\tau}(h')), \text{ and } w(h') = \omega(\hat{\tau}(h'))b(h', \hat{\tau}(h')) \geq \omega(\hat{\tau}(h))b(h', \hat{\tau}(h)).
\]

Combining these two inequalities, we have

\[
\frac{b(h, \hat{\tau}(h'))}{b(h, \hat{\tau}(h))} \leq \frac{\omega(\hat{\tau}(h))}{\omega(\hat{\tau}(h'))} \frac{b(h', \hat{\tau}(h'))}{b(h', \hat{\tau}(h))},
\]

Let \( \tau = \hat{\tau}(h) \) and \( \tau' = \hat{\tau}(h') \). Since \( \hat{\tau} \) has an inverse function \( \hat{h} \), above inequality is equivalent to

\[
\frac{b(\hat{h}(\tau), \tau')}{b(\hat{h}(\tau), \tau)} \leq \frac{\omega(\tau)}{\omega(\tau')} \leq \frac{b(\hat{h}(\tau'), \tau')}{b(\hat{h}(\tau'), \tau)}.
\]

By taking log on both sides and dividing by \( \tau' - \tau \),

\[
\frac{\log b(\hat{h}(\tau), \tau') - \log b(\hat{h}(\tau), \tau)}{\tau' - \tau} \leq \frac{-(\log \omega(\tau') - \log \omega(\tau))}{\tau' - \tau} \leq \frac{\log b(\hat{h}(\tau'), \tau') - \log b(\hat{h}(\tau'), \tau)}{\tau' - \tau}.
\]

As \( \tau' - \tau \to 0 \), we have

\[
\frac{d\log \omega(\tau)}{d\tau} = -\frac{\partial \log b(\hat{h}(\tau), \tau)}{\partial \tau},
\]

which is the equation (9).

Now consider the task production. For notational convenience, we introduce

\[
H(h, \tau) = \left[ \alpha_h(\tau)(b(h, \tau)l(h))^{\sigma_e-1} + \alpha_e(\tau) \left( \int_0^{N_e} e(k, \tau)^{\sigma_e} dk \right)^{\sigma_e-1} \right]^{\frac{\sigma_e}{\sigma_e-1}}
\]

From

\[
\max p(\tau)T(\tau) - \int_h w(h)l(h, \tau)dh - \int_0^{N_e} p_e(k)s(k, \tau)dk - \int_0^{N_e} p_e(k)e(k, \tau)dk,
\]
we have the following first order conditions:

\[ w(h) \geq \alpha_h(\tau)p(\tau)^{\frac{1}{\sigma_s}}H(h, \tau)^{\frac{\sigma_s - \sigma_e}{\sigma_e \sigma_s}} l(h)^{-\frac{1}{\sigma_e}} b(h, \tau), \]  

(C.1)

\[ p_s(k) = \alpha_s(\tau)p(\tau)^{\frac{1}{\sigma_s}}H(h, \tau)^{\frac{\sigma_s - \sigma_e}{\sigma_e \sigma_s}} \left( \int_0^{N_s} e(k, \tau)^{\nu_e} \frac{\sigma_s - 1 - \nu_e \sigma_e}{\nu_e \sigma_s} e(k, \tau)^{\nu_e - 1}, \right) \]

(C.2)

\[ p_s(k) = \alpha_s(\tau)p(\tau)^{\frac{1}{\sigma_s}} \int_0^{N_s} s(k, \tau)^{\nu_s} \frac{\sigma_s - 1 - \nu_s \sigma_s}{\nu_s \sigma_s} s(k, \tau)^{\nu_s - 1}, \]  

(C.3)

Since \( p_s(k) = q_s/\nu_s \) and \( p_e(k) = q_e/\nu_e \) from capital provider’s problem, we get

\[ p(\tau) = \left\{ \alpha_h(\tau)^{\sigma_e} \omega(\tau)^{1-\sigma_e} + \alpha_e(\tau)^{\sigma_e} \left( \frac{q_e}{\nu_e N_e^{\nu_e}} \right)^{1-\sigma_e} \right\}^{\frac{1-\sigma_s}{1-\sigma_e}} + \alpha_s(\tau)^{\sigma_s} \left( \frac{q_s}{\nu_s N_s^{\nu_s}} \right)^{1-\sigma_s} \]

by combining the FOCs, which is the equation (11).

Again from equation (C.1) to (C.3), the task production \( T(\tau) \) can be expressed by

\[ T(\tau) = p(\tau)^{\sigma_s - \sigma_e} \alpha_h(\tau)^{-\sigma_e} \left( \alpha_h(\tau)^{\sigma_e} \omega(\tau)^{1-\sigma_e} + \alpha_e(\tau)^{\sigma_e} \left( \frac{q_e}{\nu_e N_e^{\nu_e}} \right)^{1-\sigma_e} \right)^{\frac{\sigma_s - \sigma_e}{1-\sigma_e}} \int_h b(h, \tau) l(h, \tau) dh \]

From the labor market clearing condition and lemma 1, we have

\[ l(h, \tau) = \mu(h) \delta[\tau - \hat{\tau}(h)], \]

where \( \delta \) is a Dirac delta function. Then we have

\[ \int_h b(h, \tau) l(h, \tau) dh = \int_{\tau'} b(\hat{\tau}(\tau'), \tau') \mu(\hat{\tau}(\tau)) \delta[\tau - \tau'] \hat{\tau}'(\tau') d\tau' = b(\hat{\tau}(\tau), \tau) \mu(\hat{\tau}(\tau)) \hat{\tau}'(\tau). \]

Combining this with equation (7) and (C.4), we have

\[ \hat{\tau}'(\tau) = \frac{\gamma(\tau)p(\tau)^{\sigma_s - \sigma_e} \alpha_h(\tau)^{\sigma_s} \left( \alpha_h(\tau)^{\sigma_e} \omega(\tau)^{1-\sigma_e} + \alpha_e(\tau)^{\sigma_e} \left( \frac{q_e}{\nu_e N_e^{\nu_e}} \right)^{1-\sigma_e} \right)^{\frac{\sigma_s - \sigma_e}{1-\sigma_e}} Y}{\omega(\tau)^{\sigma_e} b(\hat{\tau}(\tau), \tau) \mu(\hat{\tau}(\tau))}, \]

which is the equation (10).

**Proof of lemma 3** In steady state, if it exists, \( r = \pi_s/\eta_s = \pi_e/\eta_e = \rho \) from the Euler equation (17). Then \( \dot{X}/X = 0 \) for \( X = C, E, S, N_e, \) and \( N_s \) follow from usual argument. What we need to show is that there exist \( N_s \) and \( N_e \) that satisfy \( \pi_s/\eta_s = \pi_e/\eta_e = \rho \).

We start with the following lemma.
Lemma 4 Fix $p(\tau)$ and $\hat{h}(\tau)$. There exists a pair $(\nu_s, \nu_e) \in (0,1) \times (0,1)$ such that $s(\tau)$ is strictly decreasing in $N_s$ and $e(\tau)$ is strictly decreasing in $N_e$.

Proof Combining equation (C.1) to (C.3) (FOCs), we have

$$s(\tau) = N_s^{-1} N_s^{\varphi_s(\sigma_s-1)} \left( \frac{q_s}{\nu_s} \right)^{-\sigma_s} \alpha_s(\tau)^{\sigma_s} \alpha_{\hat{h}(\tau)}^{-\frac{d}{1-\sigma_e}} b(\hat{h}(\tau), \tau) \mu(\hat{h}(\tau)) \hat{h}'(\tau)$$

$$\times \left[ \left( p(\tau)^{1-\sigma_s} - \alpha_s(\tau)^{\sigma_s} \left( \frac{q_s}{\nu_s N_s^{\varphi_s}} \right)^{1-\sigma_s} \right)^{\frac{1-\sigma_e}{1-\sigma_s}} - \alpha_e(\tau)^{\sigma_e} \left( \frac{q_e}{\nu_e (N_s n_{es})^{\varphi_e}} \right)^{1-\sigma_e} \right]^{\frac{\sigma_e}{1-\sigma_e}},$$

(C.5)

and

$$e(\tau) = N_e^{-1} N_e^{\varphi_e(\sigma_e-1)} \left( \frac{q_e}{\nu_e} \right)^{-\sigma_e} \alpha_e(\tau)^{\sigma_e} \alpha_{\hat{h}(\tau)}^{-\frac{d}{1-\sigma_e}} b(\hat{h}(\tau), \tau) \mu(\hat{h}(\tau)) \hat{h}'(\tau)$$

$$\times \left[ \left( p(\tau)^{1-\sigma_s} - \alpha_s(\tau)^{\sigma_s} \left( \frac{q_s}{\nu_s (N_e n_{es})^{\varphi_s}} \right)^{1-\sigma_s} \right)^{\frac{1-\sigma_e}{1-\sigma_s}} - \alpha_e(\tau)^{\sigma_e} \left( \frac{q_e}{\nu_e N_e^{\varphi_e}} \right)^{1-\sigma_e} \right],$$

(C.6)

where $n_{es} := N_e / N_s$.

From equation (C.5) and (C.6), we can express

$$\frac{\partial \log s(\tau)}{\partial N_s} = -\frac{1}{N_s} + s_1(\tau; \varphi_s),$$

(C.7)

$$\frac{\partial \log e(\tau)}{\partial N_e} = -\frac{1}{N_e} + e_1(\tau; \varphi_e),$$

(C.8)

and it’s straightforward to check that $\lim_{\varphi_s \downarrow 0} |s_1(\tau; \varphi_s)| = 0$, $\lim_{\varphi_e \downarrow 0} |e_1(\tau; \varphi_e)| = 0$, and $\partial s_1 / \partial \varphi_s > 0$, $\partial e_1 / \partial \varphi_e > 0$. This implies that there should be $0 < \nu_s < 1$ and $0 < \nu_e < 1$ which make $s(\tau)$ strictly decreasing in $N_s$ and $e(\tau)$ strictly decreasing in $N_e$.

Lemma 5 Fix $p(\tau)$ and $\hat{h}(\tau)$. With $\nu_e$ and $\nu_s$ close to one, we have the following:

$$\lim_{N_s \to 0} s(\tau) = \infty, \quad \lim_{N_e \to 0} e(\tau) = \infty, \quad \lim_{N_s \to \infty} s(\tau) = 0, \quad \lim_{N_e \to \infty} e(\tau) = 0.$$

Proof By substituting $\nu_e = 1$ and $\nu_s = 1$ (and hence $\varphi_e = 1 - \nu_e / \nu_e = 0$ and $\varphi_s = 1 - \nu_s / \nu_s = 0$),
0) into equation (C.5) and (C.6), we have

\[
s(\tau) = N_s^{-1} \left( \frac{q_s}{\nu_s} \right)^{-\sigma_s} \alpha_s(\tau)^{\sigma_s} \alpha_h(\tau)^{-\frac{\sigma_e}{\sigma_e}} b(\hat{h}(\tau), \tau) \mu(\hat{h}(\tau)) \hat{h}'(\tau)
\]

\[
\times \left[ \left( p(\tau)^{1-\sigma_s} - \alpha_s(\tau)^{\sigma_s} \left( \frac{q_s}{\nu_s} \right)^{1-\sigma_s} \right)^{1-\sigma_s} - \alpha_e(\tau)^{\sigma_e} \left( \frac{q_e}{\nu_e} \right)^{1-\sigma_e} \right]^{\frac{\sigma_s-\sigma_e}{\sigma_s-\sigma_e}}
\]

\[
\times \left( p(\tau)^{1-\sigma_s} - \alpha_s(\tau)^{\sigma_s} \left( \frac{q_s}{\nu_s} \right)^{1-\sigma_s} \right)^{1-\sigma_s} - \alpha_e(\tau)^{\sigma_e} \left( \frac{q_e}{\nu_e} \right)^{1-\sigma_e}
\]

(C.9)

and

\[
e(\tau) = N_e^{-1} \left( \frac{q_e}{\nu_e} \right)^{-\sigma_e} \alpha_e(\tau)^{\sigma_e} \alpha_h(\tau)^{-\frac{\sigma_e}{\sigma_e}} b(\hat{h}(\tau), \tau) \mu(\hat{h}(\tau)) \hat{h}'(\tau)
\]

\[
\times \left[ \left( p(\tau)^{1-\sigma_s} - \alpha_s(\tau)^{\sigma_s} \left( \frac{q_s}{\nu_s} \right)^{1-\sigma_s} \right)^{1-\sigma_s} - \alpha_e(\tau)^{\sigma_e} \left( \frac{q_e}{\nu_e} \right)^{1-\sigma_e} \right]^{\frac{\sigma_s-\sigma_e}{\sigma_s-\sigma_e}}
\]

(C.10)

The result is straightforward from equation (C.9) and (C.10).

Since \( \pi_e \) and \( \pi_s \) are proportional to integration of \( s(\tau) \) and \( e(\tau) \), lemma 4 and 5 imply the existence of unique steady state under some \( \nu_e \) and \( \nu_s \) large enough, fixing static equilibrium.

Note that both \( \hat{h} \) and \( \mu(\hat{h})d\hat{h} \) are bounded above by assumption and boundary conditions, and \( p(\tau) \) is also bounded as \( \int_{\tau} \gamma(\tau)p(\tau)^{1-\epsilon}d\tau = 1 \). Hence, the existence follows when \( \pi_e \) and \( \pi_s \) are continuous in \( N_e \) and \( N_s \) even when considering changes in static equilibrium. Recall that \( p(\tau) \) and \( \hat{h}(\tau) \) could be obtained from the system of differential equations (9) to (11). Since all functions in equation (9) to (11) are differentiable, \( \pi_e \) and \( \pi_s \) are also continuous in \( N_e \) and \( N_s \) and the desired result follows.

Intuitively, large \( \nu_e \) and \( \nu_s \) mean small returns to introducing additional variety, in turn, meaning decreasing rate of return. To see this intuition more clearly, recall that the task production function is given by

\[
T(\tau) = \left\{ \alpha_h(\tau) \left( b(\hat{h}(\tau), \tau) \mu(\hat{h}(\tau)) \hat{h}'(\tau) \right)^{\frac{\sigma_e-1}{\sigma_e}} + \alpha_e(\tau)N_e^{\frac{\sigma_e-1}{\sigma_e}} e(\tau)^{\frac{\sigma_e-1}{\sigma_e}} \right\}^{\pi_e(\sigma_s-1)}
\]

\[
\times \left[ \left( \frac{q_s}{\nu_s} \right)^{\sigma_s} \left( \frac{q_e}{\nu_e} \right)^{\sigma_e} \right]^{\frac{\sigma_s-1}{\sigma_s}}
\]

(C.11)

as \( s(k, \tau) = s(\tau) \) and \( e(k, \tau) = e(\tau) \) in equilibrium. The production is homogeneous of degree one in labor, \( N_e \) and \( N_s \) when \( \nu_e \to 1 \) and \( \nu_s \to 1 \). Since labor is fixed component,
the production features strict concavity along $N_e$ and $N_s$, meaning decreasing returns to scale in terms of total varieties.

The second part of lemma (3) is when $\sigma_e = \sigma_s = 1$. In this case,

\[
p(\tau)T(\tau) = \frac{\omega(\tau)b(\hat{h}(\tau), \tau)\mu(\hat{h}(\tau))h'(\tau)}{\alpha_h(\tau)},
\]

(C.12)

\[
s(\tau) = \frac{\nu_s\alpha_s(\tau)p(\tau)T(\tau)}{N_sq_s},
\]

(C.13)

\[
e(\tau) = \frac{\nu_e\alpha_e(\tau)p(\tau)T(\tau)}{N_eq_e}.
\]

(C.14)

Combining the FOCs, $T(\tau)$ satisfies

\[
p(\tau)T(\tau) = p(\tau)\left(\frac{1}{\alpha_h(\tau)}\right)^{\Psi_{es}(\tau)} \left(\frac{N_e}{N_s}\right)^{\Psi_e(\tau)} B(\tau),
\]

(C.15)

where $\kappa(\tau) := \left(\frac{\alpha_s(\tau)}{q_s} \frac{\alpha_e(\tau)}{q_e}\right) \frac{\alpha_s(\tau)}{\alpha_h(\tau)}$, $\Psi_{es}(\tau) := \frac{1-\nu_s}{\nu_s} \alpha_s(\tau) + \frac{1-\nu_e}{\nu_e} \alpha_e(\tau)$, $\Psi_e(\tau) := \frac{1-\nu_e}{\nu_e} \alpha_e(\tau)$, and $B(\tau) := b(\hat{h}(\tau), \tau)\mu(\hat{h}(\tau))h'(\tau)$ are introduced to simplify notation.

From equation (C.13) and (C.14), it is apparent that $s(\tau)$ and $e(\tau)$ are decreasing in $N_s$ and $N_e$ respectively when $\Psi_{es}(\tau) < 1$, which is a condition given in lemma 3.

\[\blacksquare\]

**Proof of proposition 1 (job polarization)** Substituting $p(\tau)$ out from equation (9) to (11), we have

\[
\hat{h}'(\tau) = \frac{\gamma(\tau)\alpha_h(\tau)1-\alpha_h(\tau)(1-\epsilon)Y}{b(\hat{h}(\tau), \tau)\mu(\hat{h}(\tau))\omega(\tau)1-\alpha_h(\tau)(1-\epsilon)} \left[ \frac{q_sN_s^{-1-\nu_s/\nu_s}}{\nu_s\alpha_s(\tau)} \left(\frac{q_eN_e^{-1-\nu_e/\nu_e}}{\nu_e\alpha_e(\tau)}\right)^{\alpha_e(\tau)} \right]^{1-\epsilon}
\]

(C.16)

\[
\frac{d\log\omega(\tau)}{d\tau} = -\frac{\partial\log b(\hat{h}(\tau), \tau)}{\partial\tau}
\]

(C.17)

First, we show $\hat{h}_1$ and $\hat{h}_2$ has to cross at least once. Suppose there is no crossing. Since $\hat{h}_1(0) = \hat{h}_2(0)$ and $\hat{h}_1(\tau) = \hat{h}_2(\tau)$, we have

\[
\left(\frac{\omega_1(0)}{\omega_2(0)}\right)^{1-\alpha_h(0)(1-\epsilon)} = \frac{\hat{h}'_1(0)}{\hat{h}'_2(0)} \left(\frac{q_{e1}}{q_{e2}}\right)^{(1-\epsilon)\alpha_e(0)},
\]

(C.18)

\[
\left(\frac{\omega_1(\tau)}{\omega_2(\tau)}\right)^{1-\alpha_h(\tau)(1-\epsilon)} = \frac{\hat{h}'_1(\tau)}{\hat{h}'_2(\tau)} \left(\frac{q_{e1}}{q_{e2}}\right)^{(1-\epsilon)\alpha_e(\tau)}.
\]

(C.19)
from equation (C.16). Combining,

\[
\left(\frac{\omega_1(\bar{\tau})}{\omega_2(\bar{\tau})}\right)^{1-\alpha_h(0)(1-\epsilon)} \left(\frac{\omega_1(\bar{\tau})}{\omega_2(\bar{\tau})}\right)^{(\alpha_h(0)-\alpha_h(\bar{\tau}))(1-\epsilon)} = \frac{\hat{h}_2'(\bar{\tau})/\hat{h}_2'(0)}{\hat{h}_1'(\bar{\tau})/\hat{h}_1'(0)}
\]

(C.20)

Since \(\hat{h}(\tau)\) is strictly monotone and continuous, with no crossing on entire \((0, \bar{\tau})\), we have to have either (i) \(\hat{h}_2'(\bar{\tau})/\hat{h}_2'(0) < \hat{h}_1'(\bar{\tau})/\hat{h}_1'(0)\) and \(\hat{h}_1(\tau) < \hat{h}_2(\tau)\) for \(\tau \in (0, \bar{\tau})\), or (ii) \(\hat{h}_2'(\bar{\tau})/\hat{h}_2'(0) > \hat{h}_1'(\bar{\tau})/\hat{h}_1'(0)\) and \(\hat{h}_1(\tau) > \hat{h}_2(\tau)\) for \(\tau \in (0, \bar{\tau})\). However, from equation (C.17) and log supermodularity of \(b(h, \tau)\), we have \(\omega_1(\bar{\tau})/\omega_1(0) > \omega_2(\bar{\tau})/\omega_2(0)\) with \(\hat{h}_1(\tau) < \hat{h}_2(\tau)\). With small enough \(\alpha_h(\bar{\tau})\), \((\omega_1(\bar{\tau})/\omega_2(\bar{\tau}))(\alpha_h(0)-\alpha_h(\bar{\tau}))(1-\epsilon)\) goes close to one, and hence equation (C.20) contradicts log supermodularity of \(b(h, \tau)\).

Second, we show that when \(\hat{h}_1(\tau)\) and \(\hat{h}_2(\tau)\) cross at any three points \(\tau_a < \tau_b < \tau_c\), we have \(\hat{h}_1'(\tau_a)/\hat{h}_2'(\tau_a) < \hat{h}_2'(\tau_a)/\hat{h}_2'(\tau_b)\) with \(\hat{h}_2(\tau) > \hat{h}_1(\tau)\) for \(\tau \in (\tau_a, \tau_b)\) and \(\hat{h}_1'(\tau_c)/\hat{h}_2'(\tau_c) > \hat{h}_2'(\tau_c)/\hat{h}_2'(\tau_b)\) with \(\hat{h}_1(\tau) > \hat{h}_2(\tau)\) for \(\tau \in (\tau_b, \tau_c)\).

From equilibrium condition (C.16),

\[
\left(\frac{\omega_1(\tau_a)}{\omega_2(\tau_a)}\right)^{1-\alpha_h(\tau_a)(1-\epsilon)} \left(\frac{\omega_1(\tau_a)}{\omega_2(\tau_a)}\right)^{(\alpha_h(\tau_a)-\alpha_h(\tau_b))(1-\epsilon)} = \frac{\hat{h}_2'(\tau_a)/\hat{h}_2'(\tau_a)}{\hat{h}_1'(\tau_a)/\hat{h}_1'(\tau_a)} \left(\frac{\bar{q}_e}{\bar{q}_e^2}\right)^{(1-\epsilon)(\alpha_e(\tau_a)-\alpha_e(\tau_a))}
\]

(C.21)

\[
\left(\frac{\omega_1(\tau_c)}{\omega_2(\tau_c)}\right)^{1-\alpha_h(\tau_c)(1-\epsilon)} \left(\frac{\omega_1(\tau_c)}{\omega_2(\tau_c)}\right)^{(\alpha_h(\tau_c)-\alpha_h(\tau_b))(1-\epsilon)} = \frac{\hat{h}_2'(\tau_c)/\hat{h}_2'(\tau_c)}{\hat{h}_1'(\tau_c)/\hat{h}_1'(\tau_c)} \left(\frac{\bar{q}_e}{\bar{q}_e^2}\right)^{(1-\epsilon)(\alpha_e(\tau_c)-\alpha_e(\tau_c))}
\]

(C.22)

With small enough \(\alpha_h'(\tau)\), these equations are approximated to

\[
\left(\frac{\omega_1(\tau_a)}{\omega_2(\tau_a)}\right)^{1-\alpha_h(\tau_a)(1-\epsilon)} \approx \frac{\hat{h}_2'(\tau_a)/\hat{h}_2'(\tau_a)}{\hat{h}_1'(\tau_a)/\hat{h}_1'(\tau_a)} \left(\frac{\bar{q}_e}{\bar{q}_e^2}\right)^{(1-\epsilon)(\alpha_e(\tau_a)-\alpha_e(\tau_a))}
\]

(C.23)

\[
\left(\frac{\omega_1(\tau_c)}{\omega_2(\tau_c)}\right)^{1-\alpha_h(\tau_c)(1-\epsilon)} \approx \frac{\hat{h}_2'(\tau_c)/\hat{h}_2'(\tau_c)}{\hat{h}_1'(\tau_c)/\hat{h}_1'(\tau_c)} \left(\frac{\bar{q}_e}{\bar{q}_e^2}\right)^{(1-\epsilon)(\alpha_e(\tau_c)-\alpha_e(\tau_c))}
\]

(C.24)

The only possibility that this can hold at the same time is when \(\alpha_e(\tau_b) > \alpha_e(\tau_a)\) and \(\alpha_e(\tau_b) > \alpha_e(\tau_c)\) so that the signs of exponent term with respect to \((\bar{q}_e/\bar{q}_e^2)\) are different. Recall that \(\omega_1(\tau_b)/\omega_1(\tau_a) < \omega_2(\tau_b)/\omega_2(\tau_a)\) implies \(\hat{h}_2'(\tau_b)/\hat{h}_2'(\tau_a) > \hat{h}_1'(\tau_b)/\hat{h}_1'(\tau_a)\) from equilibrium condition (C.17) and log supermodularity of \(b(h, \tau)\). Since \(\bar{q}_e > \bar{q}_e^2\), \(\alpha_e(\tau_b) > \alpha_e(\tau_a)\), and \(\alpha_e(\tau_b) > \alpha_e(\tau_c)\), we must have \(\omega_1(\tau_b)/\omega_1(\tau_a) > \omega_2(\tau_b)/\omega_2(\tau_a)\) and \(\omega_1(\tau_c)/\omega_1(\tau_b) < \omega_2(\tau_c)/\omega_2(\tau_b)\), which implies \(\hat{h}_1(\tau) < \hat{h}_2(\tau)\) for \(\tau \in (\tau_a, \tau_b)\) and \(\hat{h}_1(\tau) > \hat{h}_2(\tau)\) for \(\tau \in (\tau_b, \tau_c)\).

The proof in the first part rules out any even number of crossings and no crossing. The second part implies they have to cross only a single time on \(\tau \in (0, \bar{\tau})\) as they
already meet at 0 and \( \bar{\tau} \). Then the result follows from the second part of proof.

**Proof of proposition 2 (the rise of software)** We firstly show that the production share of middle skill task (task 1) falls and that of high skill task (task 2) rises in response to the decline of price of equipment in a discretized model as well. To be specific, we prove the following lemma first.

**Lemma 6** Fix \( N_e \) and \( N_s \). Consider a decline of the price of equipment; \( d\log(1/q_e) > 0 \) and suppose \( \epsilon < 1 \) and assumption 2, 5, and 6. Then we have \( d\log(p_1) < 0 \) and \( d\log(p_2) > 0 \).

**Proof** From the equilibrium conditions (B.1) to (B.3),

\[
p_j = \left( \frac{\omega_j}{\alpha_{h,j}} \right)^{\alpha_{h,j}} \left( \frac{q_e}{\nu_e \alpha_{e,j}} \right)^{\alpha_{e,j}} \left( \frac{q_s}{\nu_s \alpha_{s,j}} \right)^{\alpha_{s,j}} N_e^{-\varphi_c \alpha_{e,j}} N_s^{-\varphi_s \alpha_{s,j}}, \quad \text{for } j = 0, 1, 2
\]

\[
w_{j-1}b(\hat{h}_j, j-1) = w_jb(\hat{h}_j, j), \quad \text{for } j = 1, 2
\]

\[
\frac{\omega_{j-1} \int_{\hat{h}_j-1}^{\hat{h}_j} b(h, j-1) \mu(h) dh}{\omega_j \int_{\hat{h}_j}^{\hat{h}_j+1} b(h, j) \mu(h) dh} = \frac{\alpha_{h,j-1} \gamma_{j-1}}{\alpha_{h,j} \gamma_j} \left( \frac{p_{j-1}}{p_j} \right)^{1-\epsilon}, \quad \text{for } j = 1, 2,
\]

with \( \sigma_s = \sigma_e = 1 \).

Let \( \Delta x = d\log(x) \). Then by differentiating above and using assumption 5,

\[
\Delta p_j = \alpha_{h,j} \Delta \omega_j - \alpha_{e,j} \Delta(1/q_e) \quad \text{(C.25)}
\]

\[
\Delta \omega_{j-1} = \Delta \omega_j + \Delta b(\hat{h}_j, j) - \Delta b(\hat{h}_j, j-1) \quad \text{(C.26)}
\]

\[
\Delta \omega_{j-1} - \Delta \omega_j = (1-\epsilon)(\Delta p_{j-1} - \Delta p_j) \quad \text{(C.27)}
\]

\[
\sum_{j=0}^{2} \gamma_j p_j^{1-\epsilon} \Delta p_j = 0 \quad \text{(C.28)}
\]

Eliminating \( \omega_j \)'s,

\[
\left( \frac{1}{\alpha_{h,0}} - (1-\epsilon) \right) \Delta p_0 = \left( \frac{1}{\alpha_{h,1}} - (1-\epsilon) \right) \Delta p_1 + \left( \frac{\alpha_{e,1}}{\alpha_{h,1}} - \frac{\alpha_{e,0}}{\alpha_{h,0}} \right) \Delta(1/q_e) \quad \text{(C.29)}
\]

\[
\left( \frac{1}{\alpha_{h,2}} - (1-\epsilon) \right) \Delta p_2 = \left( \frac{1}{\alpha_{h,1}} - (1-\epsilon) \right) \Delta p_1 + \left( \frac{\alpha_{e,1}}{\alpha_{h,1}} - \frac{\alpha_{e,2}}{\alpha_{h,2}} \right) \Delta(1/q_e) \quad \text{(C.30)}
\]

Since \( 1/\alpha_{h,j} > (1-\epsilon) \) for all \( j \)'s and \( \alpha_{e,j}/\alpha_{h,1} > \alpha_{e,j}/\alpha_{h,j} \) for \( j = 0, 2 \), it is easy to check that \( \Delta p_1 < 0 \) by substituting equation (C.29) and (C.30) into equation (C.28).
Substituting equation (C.29) and (C.30) into equation (C.28), we also have

$$\gamma_0 p_0^{1-\epsilon} \left( \frac{1}{\alpha_{h,2}} - \frac{(1 - \epsilon)}{\alpha_{h,0}} \right) + \gamma_2 p_2^{1-\epsilon} + \gamma_1 p_1^{1-\epsilon} \left( \frac{1}{\alpha_{h,2}} - \frac{(1 - \epsilon)}{\alpha_{h,1}} \right) \Delta p_2$$

$$+ \gamma_0 p_0^{1-\epsilon} \left[ \left( \frac{\alpha_{e,1}}{\alpha_{h,1}} - \frac{\alpha_{e,0}}{\alpha_{h,0}} \right) - \left( \frac{\alpha_{e,1}}{\alpha_{h,1}} - \frac{\alpha_{e,2}}{\alpha_{h,2}} \right) \right] \Delta (1/q_e)$$

$$- \gamma_1 p_1^{1-\epsilon} \left( \frac{\alpha_{e,1}}{\alpha_{h,1}} - \frac{\alpha_{e,2}}{\alpha_{h,2}} \right) \Delta (1/q_e) = 0$$

(C.31)

By assumption 6 and $\epsilon < 1$, we have

$$\gamma_0 p_0^{1-\epsilon} \left( \frac{1}{\alpha_{h,2}} - \frac{(1 - \epsilon)}{\alpha_{h,0}} \right) + \gamma_2 p_2^{1-\epsilon} + \gamma_1 p_1^{1-\epsilon} \left( \frac{1}{\alpha_{h,2}} - \frac{(1 - \epsilon)}{\alpha_{h,1}} \right) > 0,$$

$$\gamma_0 p_0^{1-\epsilon} \left[ \left( \frac{\alpha_{e,1}}{\alpha_{h,1}} - \frac{\alpha_{e,0}}{\alpha_{h,0}} \right) - \left( \frac{\alpha_{e,1}}{\alpha_{h,1}} - \frac{\alpha_{e,2}}{\alpha_{h,2}} \right) \right] = 0,$$

$$\gamma_1 p_1^{1-\epsilon} \left( \frac{\alpha_{e,1}}{\alpha_{h,1}} - \frac{\alpha_{e,2}}{\alpha_{h,2}} \right) > 0,$$

implying $\Delta p_2 > 0$ from equation (C.31).

Now we show that lemma 6 implies a relative increase of software variety in the new steady state. Note that the profits from providing software and equipment variety are given by

$$\pi_e = \sum_j \frac{1 - \nu}{\nu} q_e e_j \quad \text{and} \quad \pi_e = \sum_j \frac{1 - \nu}{\nu} q_s e_j.$$

From the FOC and using $p_e = q_e/\nu$ and $p_s = q_s/\nu$, demand for equipment and software for each task are $e_j = \nu_e \alpha_{e,j} p_j T_j/(q_e N_e)$ and $s_j = \nu_s \alpha_{s,j} p_j T_j/(q_s N_s)$.

From lemma 3, we know $\pi_e/\eta = \pi_s/\eta = \rho$ in any steady state equilibrium, and hence,

$$d\pi_e = (1 - \nu) \left[ (1 - \epsilon)(\alpha_{e,0} p_0^{-\epsilon} dp_0 + \alpha_{e,1} p_1^{-\epsilon} dp_1 + \alpha_{e,2} p_2^{-\epsilon} dp_2)Y \right. \nonumber$$

$$\left. + \left( \sum_j \alpha_{e,j} p_j^{1-\epsilon} \right) dY - \frac{1}{N_e} \sum_j \alpha_{e,j} p_j^{1-\epsilon} Y dN_e \right] = 0$$

$$d\pi_s = (1 - \nu) \left[ (1 - \epsilon)(\alpha_{s,0} p_0^{-\epsilon} dp_0 + \alpha_{s,1} p_1^{-\epsilon} dp_1 + \alpha_{s,2} p_2^{-\epsilon} dp_2)Y \right.$$

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\[ + \left( \sum_{j} \alpha_{s,j} p_{j}^{1-\varepsilon} \right) dY - \frac{1}{N_s} \sum_{j} \alpha_{e,j} p_{j}^{1-\varepsilon} Y dN_s \] = 0

Combining,

\[
(1 - \varepsilon) \left[ (\alpha_{e,1} - \alpha_{s,1})p_{1}^{1-\varepsilon} dp_{1} + (\alpha_{e,2} - \alpha_{s,2})p_{2}^{1-\varepsilon} dp_{2} \right]
\]

\[
= \sum_{j} \alpha_{e,j} p_{j}^{1-\varepsilon} \left[ \frac{dN_e}{N_e} - \frac{dY}{Y} \right] - \sum_{j} \alpha_{s,j} p_{j}^{1-\varepsilon} \left[ \frac{dN_s}{N_s} - \frac{dY}{Y} \right]
\]

\[
= \sum_{j} \alpha_{s,j} p_{j}^{1-\varepsilon} \left[ \frac{dN_e - dN_s}{N_s} - \left( 1 - \frac{N_e}{N_s} \right) \frac{dY}{Y} \right] < 0,
\]

where the last equality is from no arbitrage condition (16) \( \frac{N_e}{N_s} = \sum \frac{\alpha_{s,j} \gamma_{j} p_{j}^{1-\varepsilon}}{\sum \alpha_{e,j} \gamma_{j} p_{j}} \), and the inequality is from lemma 6 and assumption 6.

Hence, we have

\[ dN_s > dN_e + (N_e - N_s) \frac{dY}{Y} \cdot \]

Since decrease in the price of equipment raise the level of production, we have \( dY/Y > 0 \). Hence, with the condition given in this proposition \( (N_e \geq N_s), (N_e - N_s) dY/Y \geq 0 \) and so \( dN_s > dN_e \). Finally, since \( N_e \geq N_s \), we have

\[ dN_s/N_s > dN_e/N_e, \]

which was to be shown.

**Proof of proposition 3 (skill demand reversal)** Suppose they cross at least once. It means that we have at least three points \( \tau_{a} < \tau_{b} < \tau_{c} \) such that \( h_1(\tau_{a}) = h_2(\tau_{a}), h_1(\tau_{b}) = h_2(\tau_{b}) \), and \( h_1(\tau_{c}) = h_2(\tau_{c}) \). Then, we have

\[
\left( \frac{\omega_{1}(\tau_{b})/\omega_{1}(\tau_{a})}{\omega_{2}(\tau_{b})/\omega_{2}(\tau_{a})} \right)^{1-\alpha_{h}(\tau_{a})(1-\varepsilon)} \left( \frac{\omega_{1}(\tau_{b})}{\omega_{2}(\tau_{b})} \right)^{(\alpha_{h}(\tau_{a})-\alpha_{h}(\tau_{b}))(1-\varepsilon)} = \frac{h_{2}'(\tau_{b})/h_{2}'(\tau_{a})}{h_{1}'(\tau_{b})/h_{1}'(\tau_{a})} \left( \frac{N_{s2}}{N_{s1}} \right) \varphi_{s}(1-\varepsilon)(\alpha_{s}(\tau_{b})-\alpha_{s}(\tau_{a})) \]

\( (C.32) \)

\[
\left( \frac{\omega_{1}(\tau_{c})/\omega_{1}(\tau_{b})}{\omega_{2}(\tau_{c})/\omega_{2}(\tau_{b})} \right)^{1-\alpha_{h}(\tau_{b})(1-\varepsilon)} \left( \frac{\omega_{1}(\tau_{c})}{\omega_{2}(\tau_{c})} \right)^{(\alpha_{h}(\tau_{b})-\alpha_{h}(\tau_{c}))(1-\varepsilon)} = \frac{h_{2}'(\tau_{c})/h_{2}'(\tau_{b})}{h_{1}'(\tau_{c})/h_{1}'(\tau_{b})} \left( \frac{N_{s2}}{N_{s1}} \right) \varphi_{s}(1-\varepsilon)(\alpha_{s}(\tau_{c})-\alpha_{s}(\tau_{b})) \]

\( (C.33) \)

where \( \varphi_{s} \equiv (1 - \nu_{s})/\nu_{s} \).

With small enough \( \alpha_{h}(\tau) \), above equations can be approximated to

\[
\left( \frac{\omega_{1}(\tau_{b})/\omega_{1}(\tau_{a})}{\omega_{2}(\tau_{b})/\omega_{2}(\tau_{a})} \right)^{1-\alpha_{h}(\tau_{a})(1-\varepsilon)} \frac{h_{1}'(\tau_{b})/h_{1}'(\tau_{a})}{h_{2}'(\tau_{b})/h_{2}'(\tau_{a})} \approx \left( \frac{N_{s2}}{N_{s1}} \right) \varphi_{s}(1-\varepsilon)(\alpha_{s}(\tau_{b})-\alpha_{s}(\tau_{a})) \]

\( (C.34) \)
Again, since matching function is continuous and monotone, and \( b(h, \tau) \) is log supermodular, signs of log of LHS in both equation (C.34) and (C.35) should be different. However, since \( \alpha_s(\tau) \) is strictly increasing, signs of log of RHS in equation (C.34) and (C.35) are same, which is contradiction.

Finally, to show \( \hat{h}_2(\tau) < \hat{h}_1(\tau) \) for \( \tau \in (0, \tau) \), recall that equilibrium condition (C.16) implies

\[
\left( \frac{\omega(\bar{\tau})/\omega(0)}{\omega(\bar{\tau})/\omega(0)} \right)^{1-\alpha_s(\bar{\tau})(1-\varepsilon)} \frac{\hat{h}'(\bar{\tau})/\hat{h}'(0)}{\hat{h}'(\bar{\tau})/\hat{h}'(0)} = \left( \frac{N_{s2}}{N_{s1}} \right)^{\varphi_s(1-\varepsilon)(\alpha_s(\bar{\tau})-\alpha_s(0))} \]

(C.36)

Since \( (1-\varepsilon)(\alpha_s(\bar{\tau})-\alpha_s(0)) > 0 \) and \( N_{s2} > N_{s1} \), we have to have \( \omega(\bar{\tau})/\omega(0) > \omega(\bar{\tau})/\omega(0) \), which implies \( \hat{h}_2(\tau) > \hat{h}_1(\tau) \).

**D Numerical examples**

To illustrate the comparative statics, we provide some numerical examples. For this example, we set:

\[
b(h, \tau) = h - \tau, \quad M(h) = \frac{1-h^{-a}}{1-h^{-a}}, \quad \gamma(\tau) = 1, \quad \alpha_s(\tau) = -2.5(\tau - .5)^2 + .6, \quad \text{and} \quad \alpha_s(\tau) = .3\tau + .025.
\]

For the parameter values, we use \( \bar{\tau} = 1, \bar{h} = 4, \bar{d}\tau = .005, \alpha = 2.5, \varepsilon = 0.7, \nu_s = 0.65, \nu_s = .8, \eta_s = \eta_e = q_s = q_e = 1, \theta = 1, \) and \( \rho = .03 \).

In the inner loop, we solve static equilibrium given \( N_s \) and \( N_e \). The equilibrium assignment function is computed from equation (9) to (11). Specifically, we use \( \hat{h}(0) = 1 \) and guess \( \hat{h}'(0) \) and \( \omega(0) \). With the guess, differential equation is solved using finite difference method. We iterate until \( \hat{h}(1) = 4 \) and \( \int p(\tau)^{1-\varepsilon}d\tau = 1 \) using Gauss-Newton method.

Then in the outer loop, we search for \( N_s \) and \( N_e \) that equate \( \pi_s/\eta_s = \pi_e/\eta_e = \rho \), again using Gauss-Newton method.

Factor intensities and the equilibrium assignment function in this example are shown in figure A1. The equipment intensity \( \alpha_e(\tau) \) is increasing on \( \tau \in [0, 0.5] \) and decreasing on \( \tau \in [0.5, 1] \), while the software intensity \( \alpha_s(\tau) \) is increasing from 0 to 1. We can also see that the equilibrium assignment function \( \hat{h}(\tau) \) is strictly increasing on
Now we compare equilibrium with $q_e = 1$ and $q_e = 0.2$ in figure A2. The assignment function in the original equilibrium (with $q_e = 1$), in the static equilibrium (with $q_e = 0.1$) and the new steady state (with $q_e = 0.2$) are depicted in figure A2(a). As expected from proposition 1 through proposition 3, we see that the assignment function in the static equilibrium (blue line) cross with the original assignment function (black line) at the middle of $\tau$. The assignment function in the steady state (red line) is generally located above the assignment function in static equilibrium (blue line).

To see the changes in the employment structure more clearly, we also plot changes in the employment share by skill percentile in figure A2(b), similar to the graph shown in figure 2.1(a). To be specific, the horizontal axis shows the tasks ($\hat{\tau}(h)$) corresponding to each percentile in the skill distribution $\mathcal{M}$, and the vertical axis shows the changes in the employment share of those tasks from the original equilibrium to new static equilibrium (blue line) and from the new static equilibrium to new steady state (red line). For example, the first two points on the horizontal axis is two task $\hat{\tau}_1(h_1)$ and $\hat{\tau}_1(h_2)$ where $\hat{\tau}_1$ represents the original (inverse) assignment function and $h_1 = 1$ and $h_2 = \mathcal{M}^{-1}(1)$. Then the first point on the blue line is difference between $\mathcal{M}(\hat{h}_2(\hat{\tau}_1(h_1))) - \mathcal{M}(\hat{h}_2(\hat{\tau}_1(h_2)))$ and $\mathcal{M}(h_2) - \mathcal{M}(h_1)$, where $\hat{h}_2$ is the assignment function in the static equilibrium corresponding to $q_e = .2$.

For the relative size of software variety to equipment variety, it was initially .74 in the original equilibrium, and increases to .77 in the new steady state, which is about 6% increase.

Fig. A1: Factor intensities and assignment function

(a) Intensities $\alpha(\tau)$

(b) Assignment function $\hat{h}(\tau)$
Fig. A2: Equilibrium comparison with $q_e = 1$ and $q_e = .2$

(a) Assignment function $\hat{h}(\tau)$

(b) Changes in employment by skill percentile

E Data construction for section 5

This section describes data used in section 5. We test two relations: relative employment growth against relative price growth and relative innovation growth against relative price growth.

For relative employment by industry, we use a ratio of employment of routine occupations and employment of cognitive occupations. Routine occupations include machine operators, office and sales, mechanics, construction and production, and transportation occupations. Cognitive occupations are management, professionals, and technicians. The level of employment is obtained from Census 1980, 1990, and 2000, and American Community Survey (ACS) 2010, received from IPUMS. We made a concordance between consistent industry code `ind1990` and `indnaics` using employment in Census 2000. Then employment by `indnaics` is merged into 61 BEA industry code based on a concordance between BEA industry code and NAICS.

The price of equipment and software by industry is from Section 2 of Fixed Asset Table from BEA. The price index is constructed by dividing nominal investment by real investment. We use private non-residential equipment investment by industry for the benchmark, although other series (e.g. industrial equipment) also give similar results.

For growth of software innovation, we use log difference of own account software investment by industry, which captures software investment made in-house by firms. We believe this as a good proxy for software innovation, as in-house software investment is made to develop new software for firm’s production process.
It is not straightforward to measure R&D for equipment related innovation from industry level data, as BEA records R&D expenditures only by sources of funds. We think that R&D expenditures funded by equipment producing industries are likely to be used for equipment related innovation, but they should be only a subset of total equipment related innovations. It is also likely that most of these expenditures are used by equipment producing industries, not others, which makes it difficult to capture industry variation. Therefore, we use total R&D expenditures other than software as a benchmark series for $N_e$, and examine robustness using many different combinations of R&D data. All combinations, including a case with own-account software only, show similar positive relation against relative price.

F Software embodied in equipment

We highlight the different use of software and equipment by occupations. However, the software has to be integrated into the equipment to be utilized by a human. We conceptually differentiate additional software made to support works by human and software embodied in equipment in the production of equipment. The software investment highlighted in the model is the former, not including latter. One may wonder whether the software integrated with equipment from the production of equipment has a similar pattern with software investment or equipment investment.

Regarding data, the former appears as software investment in NIPA, and the latter appears as intermediate consumption of software by equipment producing industries in the Input-Output (IO) table. To investigate the pattern of two, we construct software-embodied-equipment series from IO table and then compare it with equipment investment.

To be specific, we firstly get intermediate consumption of software commodities, 511200 (software publishers) and 541511 (custom computer programming services) from a detailed Input-Output table. This information is only available for the year 1997, 2002, and 2007. Hence, we linearly inter- and extrapolate the ratio of commodities 511200 out of 511 and 541511 out of 5415 in the year 1997, 2002, and 2007 to the periods 1980 to 2014. Then by multiplying the estimated ratio to intermediate use of industry 511 and 5415, we get estimates for software intermediate in the years other than 1997, 2002, and 2007. We finally divide this by the amount of equipment investment to compare trends between two series. The resulting series is in figure A1.

As can be seen in the figure, there is no clear trend in software embodied in equipment, implying that the rise of software is phenomenon refined to the use of software by
Fig. A1: Estimated series for intermediate use of software / equipment investment

human workers, not incorporated in equipment in the stage of equipment production.