Evolutionary Internalization of Congestion Externalities When Values of Time are Unknown

Preliminary Draft

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Abstract

We consider evolutionary implementation of optimal traffic via price scheme as Sandholm [Evolutionary Implementation and Congestion Pricing, Review of Economic Studies 69 (2002), 667-689] does. However, we analyse the opposite case to Sandholm’s in which the planner knows the demand functions but he does not know the values of time of potential drivers. We show that the planner can achieve the evolutionary implementation by assuming that economy is always at social optimum. If all potential drivers have the same value of time, the analytical technique used by Sandholm is applicable. However, if the value of time is heterogeneous, it is no longer applicable and we make several additional assumptions to make the implementation possible.

JEL classification:

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1 Introduction

In this paper, I ask the following question:

Consider a congested road. When the planner does not know the values of time of road users, how can he achieve optimal traffic?

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The environment we consider is characterized by an important property that there is a large number of anonymous agents. Generally, to achieve the social optimum, we can consider a standard one shot revelation mechanism in which agents submit their messages and the planner chooses an outcome based on the messages he received. However, from a practical point of view, such one shot implementation is labor-intensive and therefore, difficult to carry out, especially in large economies like our model. Moreover, we can show by example that a dominant strategy implementation can be impossible if agents are anonymous. If it is so, a likely situation is that agents gradually learn to play the equilibrium strategy. Therefore, it is important to consider the stability of an equilibrium outcome in a mechanism or investigate whether there exists a learning process of agents which gives rise to the outcome.

In our road congestion model, what the planner needs to do to achieve the social optimum is to internalize the congestion externalities. Sandholm (2002, 2005b) deals with a case in which the planner knows the travel cost function but he does not know the demand functions of road users. Interestingly, he shows that if the planner can correctly calculate the values of the congestion externalities for each traffic volume, she can achieve the social optimum in the long-run without collecting messages. Specifically, Sandholm considers an evolutionary Nash implementation via a price scheme in which instead of collecting messages, the planner levies taxes for each action in each period and agents learn to play equilibrium strategies according to an evolutionary dynamic. This mechanism is not labor-intensive and respects the anonymity of agents. Under his price scheme, the planner just needs to internalize the congestion externalities evaluated at realized traffic in each period to make the traffic converge to the optimal one. Theoretically, the planner can create a potential game by properly changing payoff functions through a price scheme. In addition, Sandholm’s model has an advantage that we can analyze the stability of an outcome since a potential function serves as a Lyapunov function. Indeed, the definition of evolutionary implementation guarantees that the agents learn to play the socially optimal strategy.

This paper deals with a case which is opposite to the Sandholm’s one: The planner knows the demand functions of road users but he does not know the travel cost function. To implement Sandholm’s price scheme, it is necessary that the

\[\text{See Sandholm (2005b) for an example.}\]
government knows the travel cost function. This can be a problem in the context of road traffic congestion because the travel cost includes time cost, which is travel time multiplied by value of time (VOT) of the agents. Therefore, by assuming that the planner knows the travel cost function, we implicitly assume that he knows the VOT’s of each agent. But in reality, it is natural to think that the VOT’s are private information. Moreover, in his papers, it is implicitly assumed that the VOT is homogeneous. But it is not realistic to assume that a large number of agents shares the same VOT. This paper is motivated by these points. If the planner does not know the VOT’s, he no longer can calculate the value of the congestion externalities evaluated at current traffic, and therefore, he cannot carry out the Sandholm’s scheme. In addition, it will be shown that if VOT is heterogeneous, the potential game approach, which Sandholm (2002, 05b) uses, is not applicable.

In our price scheme, the planner makes an estimate for the value of congestion externalities that coincides with the true one only at social optimum. We will see that although the economy is not necessarily at equilibrium in each period, the planner can achieve the evolutionary implementation by assuming that the economy is always at equilibrium. We start from the simplest case in which the VOT is homogeneous. That is, all potential drivers have the same VOT. In this case, the potential game approach is applicable and we can construct a price scheme that achieves a social optimum with the same class of evolutionary dynamics as Sandholm (2002, 05b) employs. However, if the VOT is heterogeneous, the potential game approach is no longer applicable. Due to this technical difficulty, we focus on a specific dynamic to achieve the evolutionary implementation. In particular, we show that the projection dynamic is well-behaved under some conditions. However, we also propose a class of dynamics under which the evolutionary implementation is possible.

The rest of this paper proceeds as follows. In the next section, basic setup of the model and several definitions are stated. In Sections 3 and 4, we construct price schemes that achieve a social optimum in the long-run. In Section 3, the VOT is assumed to be homogeneous while in Section 4, it is assumed to be heterogeneous. In Section 4, a specific example is also illustrated. Section 5 concludes and discusses subjects for future research. In Appendix, proofs omitted from the main text are provided.
2 The model

We consider a city which has one suburb area and one central business district (CBD), and is populated by a unit mass of agents. All of the agents live in the suburb and, if they work, commute to the CBD. For simplicity, we assume that there is only one road which connects the suburb and the CBD. For each member of the population, a strategy space is given by $S = \{0, 1\}$ where 0 [resp. 1] represents the action of staying at home [resp. working].

Values of time (VOT) can be different among the agents, but the number of possible VOT’s in the city is finite. Let $R$ be the set of indices for the possible VOT’s. We divide the population into subpopulations according to their VOT. Let $m_r$ be the mass of subpopulation $r \in R$. We require that $\sum_{r \in R} m_r = 1$.

Let $D'_r : R_+ \rightarrow [0, m'_r]$ be the demand function of subpopulation $r$ for commuting. We assume that $D'_r (\cdot)$ is $C^1$, onto, and strictly decreasing so that the inverse demand function $\tilde{D}'_r (\cdot)$ is well-defined and strictly decreasing.

Let $Z'_r = \{ z'_r \in R_+^{|S|} : \sum_{i \in S} z'_r i = m'_r \}$ denote the set of distributions of agents in population $r$ over actions in $S$. We define $Z = \bigcup_{r \in R} Z'_r = \{ z \in R_+^{|S| \times |R|} : \forall r \in R, \sum_{i \in S} z'_r i = m'_r \}$ and $X = \{ x \in R_+^{|S|} : x_1 = \sum_{r \in R} z'_r i, z \in Z \}$. $Z$ is the set of population states while $X$ is the set of distributions of agents over actions in $S$ and hence represents aggregate states.

In each population, a marginal driver obtains the least payoff of all active drivers in that population. The payoff of marginal driver in population $r$ is given by $F'_r(z) = \tilde{D}'(z'_r) - \theta'_r T(x_1)$ where $\theta'_r$ is the VOT of population $r$, $T(\cdot)$ is the travel time function, and $x_1 = \sum_{r \in R} z'_r i$. We assume that $T(\cdot)$ is $C^2$, strictly increasing, strictly convex, and $T(0) > 0$. Generally, an agent who obtains a payoff from commuting which is higher than the marginal driver’s payoff does not necessarily drive out of equilibrium. However, to make analysis tractable, we assume that he does drive not only at equilibrium but also out of equilibrium. Since the travel cost is increasing in road utilization, there are congestion externalities. On the other hand, the payoff of staying at home is identically zero. Let $F'_0(z)$ be the payoff of an agent in population $r$ who stays at home. Then, $F'_0(z) = 0$ for all $r \in R$.

Let $\Theta' \subseteq R_+$ be the set of all possible VOT’s of population $r$. Let $\Theta$ be a subset of

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2It is not difficult to extend the model to allow the possibility of route choice.

3We denote the cardinality of a set $A$ by $|A|$.

4In Sandholm (2002, 05b), travel cost is given by $C(x_1)$ and it is assumed that the government knows $C(\cdot)$. 

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\( \prod_{r \in R} \Theta_r \). \( \Theta \) is interpreted as a type space of the population. We denote an element of \( \Theta \) by \( \theta \). In the following analysis, we assume that the government knows \( T(\cdot) \) and \( \{D'(\cdot)\}_{r \in R} \) but does not know \( \{\theta'_r\}_{r \in R} \). Moreover, due to the anonymity of the agents, the planner can observe aggregate states but cannot observe population states.

Given the above setting, we can represent a population game as \( G = (F, \theta) \) where \( F \) is the vector of \( F'_r \)'s. \(^5\) \( z \in Z \) is a Nash equilibrium of \( G \) if for all \( r \in R \),

\[
\begin{cases} 
F'_1(z) \leq F'_0(z) & \text{if } \bar{z}'_1 = 0, \\
F'_1(z) = F'_0(z) & \text{if } \bar{z}'_1 \in (0, m'), \\
F'_1(z) \geq F'_0(z) & \text{if } \bar{z}'_1 = m'. 
\end{cases}
\]

2.1 Admissible evolutionary dynamic

In our model, the agents do not necessarily behave rationally in each date and hence, the economy is not always at equilibrium. Their decision process is described by an evolutionary dynamic which is a vector field \( V : Z \to \mathbb{R}^{S \times |R|} \) defining an equation of motion \( \dot{z} = V(z) \). But instead of restricting our attention to some specific dynamic, we consider a class of dynamics that satisfy several conditions. In Section 3 where VOT is homogeneous, we consider a class of admissible evolutionary dynamics which is defined in Sandholm (2001):

**Definition 1** An evolutionary dynamic \( V : Z \to \mathbb{R}^{S \times |R|} \) is called admissible for \( G = (F, \theta) \) if it satisfies the following conditions:

(LC) \( V \) is Lipschitz continuous,

(FI) \( Z \) is forward invariant under \( \dot{z} = V(z) \),

(PC) \( V(z) \neq 0 \Rightarrow \sum_{r \in R} \text{Cov}(V', F') > 0 \),

(NC) \( V(z) = 0 \Rightarrow z \) is a Nash equilibrium of \( G \),

This class of dynamic is also considered in Sandholm (2002, 05b). Condition (LC) is a technical assumption to ensure that there exists a unique solution for each initial condition. Intuitively, this condition requires that there is no jump in changes of the aggregate states. Condition (FI) is also a technical assumption to ensure that the solution does not leave \( Z \). Condition (PC) states that if \( V(z) \neq 0 \), the sum

\(^5\)A population game is characterized by the existence of a large number of anonymous agents.
of sample covariance between $V^r$ and $F^r$ over $r \in R$ is positive. If condition (FI) holds, it follows that $\sum_{r \in R} \text{Cov}(V^r, F^r) > 0$ if and only if $F \cdot V > 0$. Moreover, since $F^r_0 = 0$, $F \cdot V = \sum_{r \in R} F^r_1 V^r_1$. If agents behave rationally, $F^r_1 V^r_1 > 0$: as long as working yields higher payoff, that is, $F^r_1 > 0$, the number of active drivers should increase. Condition (PC) requires that this relation holds only in aggregate. Therefore, if agents behave according to an admissible dynamic, it is possible out of equilibrium that the number of active drivers decrease even though working yields higher payoff. Condition (NC) says that if $z$ is a rest point of $V$ which is admissible for $G$, then $z$ is a NE of $G$. That is, if there is a profitable deviation, some agents change their strategies. As Proposition 4.3 in Sandholm (2001) shows, conditions (FI) and (PC) imply that the converse is also true. Therefore, it follows that under the conditions (FI), (PC), and (NC), $z$ is a rest point of $V$ which is admissible for $G$ if and only if $z$ is a NE of $G$. For specific examples of admissible evolutionary dynamic, see Sandholm (2005a). Although we employ this class of dynamics in the next section, we consider another class of dynamic which is stronger than this one in Section 4 where VOT is heterogeneous.

2.2 Evolutionary implementation via price scheme

We define a social choice function as a function $f : \Theta \to Z$. Also, we define a price scheme as a function $P : X \to \mathbb{R}^{|S|}$. Note that the domain of $P$ is not $Z$ since the planner can observe only aggregate states. We define a new game $G_P = (F - P, \theta)$ where the payoff from the strategy $i \in S$ for marginal agent in population $r$ is $F^r_i - P_i$. For this game, a Nash equilibrium $\bar{z} \in Z$ satisfy

$$
\begin{cases}
F_1^r(\bar{z}) - P_1(\bar{x}) \leq F_0^r(\bar{z}) - P_0(\bar{x}) & \text{if } \bar{x}_1^r = 0, \\
F_1^r(\bar{z}) - P_1(\bar{x}) = F_0^r(\bar{z}) - P_0(\bar{x}) & \text{if } \bar{x}_1^r \in (0, m^r), \\
F_1^r(\bar{z}) - P_1(\bar{x}) \geq F_0^r(\bar{z}) - P_0(\bar{x}) & \text{if } \bar{x}_1^r = m^r,
\end{cases}
$$

for all $r \in R$ where $\bar{x}_i = \sum_{r \in R} \bar{x}_i^r$ for each $i \in S$.

We consider classes of dynamics that describe a motion of population state under $G_P$. Given a type space $\Theta$ and a class $\mathcal{V}_P$ of dynamics, we say that the price scheme $P$ globally implements the social choice function $f$ in the following sense:

**Definition 2** The price scheme $P$ globally implements the social choice function $f$ under $(\Theta, \mathcal{V}_P)$ if for all $\theta \in \Theta$, $f(\theta)$ is a global attractor in $Z$ under any dynamic $V \in \mathcal{V}_P$. 

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In the next section, we consider the class of dynamics that is admissible for $G_P$, which is denoted by $V^A_P$. In Section 4, we will define another class of dynamics which is stronger than the class of admissible dynamics.

We consider the following social welfare function:

$$SW(\theta, z) = \sum_{r \in R} \left[ \int_0^{x^*_1} \tilde{D}^r(y) dy - z^*_1 \theta^r T(x_1) \right]. \quad (3)$$

This is the total payoff of the commuters.\(^6\) The government would like to implement the social choice function $f^* : \Theta \rightarrow Z$ defined by

$$f^*(\theta) \in \arg \max_{z \in Z} SW(\theta, z). \quad (4)$$

In what follows, we take a type space $\Theta$ so that $SW(\theta, \cdot)$ is strictly quasi-concave on $Z$. Then, since $Z$ is convex, $f^*(\theta)$ is nonempty and singleton for all $\theta \in \Theta$.

If the government knows $\{\theta^r\}_{r \in R}$ and can observe population states, it can calculate the optimal traffic $x^*_1 = \sum_{r \in R} z^*_1 r^r$ by maximizing (3) over $Z$ and can achieve it by levying the following Pigouvian tax:

$$P = \left( 0, T'(x^*_1) \sum_{r \in R} z^*_1 r^r \right). \quad (5)$$

where $\sum_{r \in R} z^*_1 r^r$ is the total VOT of the agents who should drive at the social optimum. However, since the government does not know $\{\theta^r\}_{r \in R}$ and can observe only aggregate states in our setting, it can calculate neither $x^*_1$ nor $\sum_{r \in R} z^*_1 r^r$.

3 Homogeneous value of time

In this section, we will study a case in which all agents have the same VOT. Hence, $|R| = 1$. For notational simplicity, we drop the superscripts in $\tilde{D}^r$ and $\theta^r$. Also, we let $Z = X = \{ x \in \mathbb{R}^{|S|}_+ : \sum_{i \in S} x_i = 1 \}$. We consider a type space $\Theta_0 \subseteq \Theta$ such that a unique social optimum is interior by assuming that $\tilde{D}(0) > \theta T(0)$ and $\tilde{D}(1) < \theta (T(1) + T'(1))$. Hence, $\Theta_0 = \left( \frac{\tilde{D}(1)}{T(1) + T'(1)}, \frac{\tilde{D}(0)}{T(0)} \right)$. If the agents behave rationally at the initial period and the equilibrium is interior (that is, if current traffic is given by the intersection of the inverse demand curve and the travel cost curve), the planner can immediately obtain the VOT as $\theta = \tilde{D}(x_1)/T(x_1)$. However, in

\(^6\)Recall that the payoff from staying at home is zero.
our model, the agents’ behaviors are given by an evolutionary dynamic, which is not always consistent with the rational behavior. Therefore, we need to consider a price scheme which leads the economy to the social optimum. Note that the socially optimal traffic $x_i^*$ satisfies $\bar{D}(x_i^*) = \theta T(x_i^*) + \theta x_i^* T'(x_i^*)$. Hence, the VOT can be represented as $\theta = \bar{D}(x_i^*)/(T(x_i^*) + x_i^* T'(x_i^*))$.

Now, consider a price scheme

$$P(x) = \left(0, \frac{\bar{D}(x_1)}{T(x_1) + x_1 T'(x_1)} x_1 T'(x_1)\right)$$

$$= \left(0, \frac{\bar{D}(x_1) \varepsilon_T(x_1)}{1 + \varepsilon_T(x_1)}\right),$$

where $\varepsilon_T(x_1) = x_1 T'(x_1)/T(x_1)$, the elasticity of travel time with respect to traffic. In each period, the estimate for $\theta$ is taken as $\bar{D}(x_1)/(T(x_1) + x_1 T'(x_1))$. Although $P_1(x)$ is not generally equal to the value of the congestion externalities evaluated at $x_1$, it gives the true value of the congestion externalities at $x_i^*$. Sandholm (2001), who takes a potential game approach, shows that if there exists a strictly concave potential function for the payoff vector $F - P$, then the maximizer of the potential function is a global attractor under any dynamic which is admissible for $G_P$. As in the Sandholm’s papers, we consider the class of dynamics which is admissible for $G_P$ here and try to construct a strictly concave potential function for $F - P^*$. For this purpose, consider the following function:

$$W(x) = \int_0^{x_1} \left(\bar{D}(r) - \theta T(r) - P_1^*(r)\right) dr. \tag{7}$$

This is clearly a potential function for $F - P^*$. Furthermore, if $P_1^*(x_1)$ is nondecreasing in $x_1$, $W(\cdot)$ is strictly concave. A sufficient condition for the price to be nondecreasing is $\varepsilon_T^*(x_1) \geq 0$. Recall that the social optimum satisfies

$$\bar{D}(x_1^*) = \theta T(x_1^*) + \frac{\bar{D}(x_1^*) \varepsilon_T(x_1^*)}{1 + \varepsilon_T(x_1^*)}. \tag{8}$$

But this is exactly the same as the first-order condition for maximization of $W$. Thus, it follows that the maximizer of $W$ is the social optimum. Therefore, we have proven the following result:

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7 A $C^1$ function $W: Z \to \mathbb{R}$ is a potential function for $F - P$ if $\frac{\partial W}{\partial x_i} = F_i - P_i$ for all $i \in S$ and $r \in R$. A potential game has a potential function for its payoff vector.

8 It follows that $\varepsilon_T^*(x_1) \geq 0$ $\Leftrightarrow$ $\varepsilon_T(x_1) \leq 1 + \frac{x_i T'(x_1)}{T(x_1)}$. Since $T$ is strictly convex, $\frac{x_i T'(x_1)}{T(x_1)} > 0$ for $x_1 > 0$. Hence, travel time can be elastic, but the condition puts an upper bound.
Proposition 1 Suppose VOT is homogeneous and \( \varepsilon_T \) is nondecreasing. Then, the price scheme \( P^* \) globally implements the efficient social choice function \( f^* \) under \((\Theta_0, \mathcal{V}^A_P)\).

Note that the planner can check the condition \( \varepsilon'_T(x_1) \geq 0 \) since he knows \( T(\cdot) \). If the condition does not hold, another trivial way is also available to the government. The equilibrium of \((F, \theta)\) is a global attractor under any admissible dynamic for \((F, \theta)\) since we can take a strictly concave potential function for \( F \) as \( \int_0^{x_1} (\tilde{D}(r) - \theta T(r)) dr \) which is maximized at the equilibrium. Therefore, if the equilibrium is interior, the government can obtain \( \theta \) by waiting until the economy goes to the equilibrium.

4 Heterogeneous value of time

In this section, we analyze a case in which VOT is heterogeneous. However, once we assume that VOT is heterogeneous, we face the difficulty which is shown in the following proposition:

Proposition 2 Suppose there exists \( z \in Z \) such that payoff is continuously differentiable at \( z \). If VOT is heterogeneous, then \( G_P \) is not a potential game for any price scheme \( P \) which respects the anonymity of the agents.

Proof. Let \(|R| \geq 2\). Suppose \( G_P \) is a potential game. Then,

\[
\frac{\partial (F^*_i - P_i)}{\partial z^s_j} = \frac{\partial (F^*_s - P_j)}{\partial z^r_i} \quad \text{for all} \quad i, j \in S \text{ and } s, r \in R. \tag{9}
\]

Therefore, \(-\theta^r T'(x_1) - \frac{\partial P_i}{\partial z^r_i} = -\theta^s T'(x_1) - \frac{\partial P_i}{\partial z^s_i} \Rightarrow \frac{\partial P_i}{\partial z^s_i} - \frac{\partial P_i}{\partial z^r_i} = (\theta^r - \theta^s) T'(x_1) \Rightarrow \frac{\partial P_i}{\partial z^s_i} \neq \frac{\partial P_i}{\partial z^r_i} \)

for \( r \neq s \). If \( P \) respects the anonymity of the agents, it can depend on \( z^r_1 \) and \( z^s_1 \) only through \( x_1 \), hence \( \frac{\partial P_i}{\partial z^r_1} = \frac{\partial P_i}{\partial z^s_1} = \frac{\partial P_i}{\partial x_1} \). Therefore, \( P \) does not respect the anonymity of the agents. Q.E.D.

Therefore, unless we use a pathological function whose derivative is not continuous at any point, the above proposition implies that the potential game approach, which we took in the case of homogeneous VOT, is not applicable here. Thus, in this section, we focus on a specific dynamic at first. In particular, we consider the projection dynamic. Also, we propose a class of strongly admissible dynamics under which the evolutionary implementation is possible. However, several conditions
imposed on that class are strong and it turns out that the projection dynamic is well-behaved even if it is not strongly admissible.

Whatever dynamic we consider, it is necessarily that we construct a price scheme such that the associated game has the social optimum as a unique equilibrium by definition of the evolutionary implementation. In the following, we try to construct such price scheme. But to make analysis tractable, we make several simplifying assumptions. First of all, we assume that VOT takes only two values (a high value or a low value). We let $R = \{H, L\}$ where $\theta^H > \theta^L$. Moreover, we also assume that the planner knows $m^H$ (that is, the share of potential drivers who have the high value of time) and $\theta^H - \theta^L$ (that is, the difference between the high value of time and the low one).\footnote{As we discuss later, if we assume that the economy is at equilibrium at initial state, we can drop the assumption that the planner knows $\theta^H - \theta^L$.}

However, he does not know $(\theta^H, \theta^L)$. The planner can observe $x_1(= z_1^H + z_1^L)$ but due to the anonymity of the agents, he cannot observe $(z_1^H, z_1^L)$.

As in the previous section, we take a type space such that a unique social optimum is interior. The interior optimality conditions are given by

\begin{align*}
\tilde{D}^H(z_1^H) &= \theta^H T(x_1^*) + (\theta^H z_1^H + \theta^L z_1^L) T'(x_1^*), \tag{10} \\
\tilde{D}^L(z_1^L) &= \theta^L T(x_1^*) + (\theta^H z_1^H + \theta^L z_1^L) T'(x_1^*), \tag{11}
\end{align*}

(10) is the condition for marginal driver who has the high value of time and (11) is the condition for marginal driver who has the low value of time.

Since the planner cannot observe $(z_1^H, z_1^L)$, he needs to estimate not only $(\theta^H, \theta^L)$ but also $z_1^L$. Let $\tilde{z}_1^L$ be the planner’s estimate for $z_1^L$. Given $x_1$, $\tilde{z}_1^L$ is taken as

$$
\tilde{z}_1^L(x_1) = \arg \min_{y \in \Gamma(x_1)} \left\{ \frac{\tilde{D}^H(x_1 - y) - \tilde{D}^L(y)}{T(x_1)} - (\theta^H - \theta^L) \right\}, \tag{12}
$$

where $\Gamma(x_1) \equiv \{ y \in [0, m^L] : \max\{0, x_1 - m^H\} \leq y \leq \min\{m^L, x_1\} \}$. Since $\frac{\tilde{D}^H(x_1 - y) - \tilde{D}^L(y)}{T(x_1)}$ is strictly increasing in $y$, $\tilde{z}_1^L(x_1)$ is uniquely determined. Moreover, it follows that $\tilde{z}_1^L(\cdot)$ is continuous and nondecreasing (see Lemma 3 in Appendix). Therefore, we can verify that $\tilde{z}_1^L(\cdot)$ is continuously differentiable almost everywhere. Since an evolutionary dynamic is normally a function of the payoffs, this will be crucial when we discuss about specific examples of the dynamic as in the next section.

\footnote{Recall that the planner can observe $x_1(= z_1^H + z_1^L)$. Hence, he can get $z_1^H$ once he knows $z_1^L$.}
Now, the planner’s estimates \((\bar{\theta}^H, \bar{\theta}^L)\) for \((\theta^H, \theta^L)\) solve
\[
\begin{align*}
\bar{D}^H(x_1 - z_1^1) &= \bar{\theta}^H T(x_1) + ((x_1 - z_1^1)\bar{\theta}^H + z_1^1\bar{\theta}^L) T'(x_1), \\
\bar{D}^L(z_1^1) &= \bar{\theta}^L T(x_1) + ((x_1 - z_1^1)\bar{\theta}^H + z_1^1\bar{\theta}^L) T'(x_1).
\end{align*}
\] (13)
At each date, the planner levies a congestion tax \(P^*_1(x) = ((x_1 - z_1^1)\bar{\theta}^H + z_1^1\bar{\theta}^L) T'(x_1)\). If an agent does not drive, he will not pay any tax, so \(P^*_0 = 0\).

Recall that we take a type space \(\Phi_0\) such that a unique social optimum is interior. We show that the game \(G_{p^*}\) has a unique equilibrium and it is the social optimum. At first, we see that \(G_{p^*}\) has a unique equilibrium in the interior of \(Z\) and it coincides with the social optimum:

**Lemma 1** For all \(\theta \in \Theta_0\), \(z\) is the social optimum iff \(z\) is an interior equilibrium of \(G_{p^*}\).

**Proof.** Let \((z^H_1, z^L_1)\) be the social optimum. Then, \(
\frac{D^H(x_1 - z_1^1) - D^L(z_1^1)}{T(x_1)} = \theta^H - \theta^L,
\)
so \(z^L_1 = z^L_1\). Therefore, the planner’s estimates \((\bar{\theta}^H, \bar{\theta}^L)\) satisfy
\[
\begin{align*}
\bar{D}^H(z_1^H) &= \bar{\theta}^H T(x_1) + P^*_1(x_1), \\
\bar{D}^L(z_1^L) &= \bar{\theta}^L T(x_1) + P^*_1(x_1),
\end{align*}
\] (14)
where \(P^*_1(x_1) = (z_1^H \bar{\theta}^H + z_1^L \bar{\theta}^L) T'(x_1)\). Hence, it follows that \((\bar{\theta}^H, \bar{\theta}^L) = (\theta^H, \theta^L)\). Then, we conclude that \((z^H_1, z^L_1)\) is an interior equilibrium of \(G_{p^*}\).

Conversely, suppose \((z^H_1, z^L_1)\) is an interior equilibrium of \(G_{p^*}\). Then, \(
\frac{D^H(x_1 - z_1^1) - D^L(z_1^1)}{T(x_1)} = \theta^H - \theta^L,
\)
so \(z_1^L = z_1^L\). Hence, the planner’s estimates \((\bar{\theta}^H, \bar{\theta}^L)\) satisfy (14) and therefore, \((\bar{\theta}^H, \bar{\theta}^L) = (\theta^H, \theta^L)\). Then, since \(P^*_1(x_1) = (z_1^H \bar{\theta}^H + z_1^L \bar{\theta}^L) T'(x_1)\), it follows that \((z^H_1, z^L_1)\) is the social optimum. \(Q.E.D.\)

From the above lemma, equilibrium of \(G_{p^*}\) uniquely exists and coincides with the social optimum in the interior of \(Z\). Thus, if \(G_{p^*}\) does not have an equilibrium on the boundary of \(Z\), \(G_{p^*}\) has the social optimum as a unique equilibrium. Let \(\varepsilon_{D_h}\) be the elasticities of \(D^h\) with respect to \(z^h_1\). The following lemma shows that it it true if \(D(\cdot)\) and \(T(\cdot)\) satisfy a certain condition:

**Lemma 2** Suppose \(\varepsilon_T(m^H + z_1^L) \leq \frac{(m^H / m^L + 1)D^L(z_1^L)}{D^H(m^H) - D^L(z_1^L)} |\varepsilon_{D_h}(z_1^L)|\) for all \(z_1^L \in [0, m^L]\) such that \(D^L(z_1^L) \leq D^H(m^H)\). Then, \(G_{p^*}\) does not have an equilibrium on the boundary for all \(\theta \in \Theta_0\).
Therefore, if we restrict our attention to trios of demand functions and travel time function that satisfy the above condition, $G_{P^{**}}$ has the social optimum as a unique equilibrium. Two remarks are in order. We needed the assumption that the planner knew $\theta^H - \theta^L$ since while we had two equilibrium conditions (10) and (11), there were three unknowns $\theta^H$, $\theta^L$, and $z_1^L$. However, if we suppose that the economy is at equilibrium at initial date, we additionally get two equilibrium conditions of no toll equilibrium, while we additionally get only one unknown, the number of active drivers who have the low (or high) value of time at the no toll equilibrium. Hence, we can drop the assumption. The second remark is that although we used the same notation $\Theta_0$ for the type spaces in homogeneous and heterogeneous cases, they are different because different conditions were imposed on $D$ and $T$ in each case.

4.1 Example: Projection Dynamic

Now that we get a price scheme such that the associated game has the social optimum as a unique equilibrium, we discuss about a specific dynamic under which the price scheme $P^{**}$ globally implements the efficient social choice function $f^*$. We argue that under some conditions, the projection dynamic is well-behaved.

The projection dynamic is given by

$$\dot{z} = \Pi_{TZ(z)} (F(z) - P(x)), \quad (15)$$

where $\Pi_X(x) = \arg\min_{y \in X} \|x - y\|$ and $TZ(z)$ is the cone of feasible directions of motion from the state $z$.\textsuperscript{11} As Lahkar and Sandholm (2008) argue, although this dynamic is discontinuous at the boundary, solution uniquely exists for all initial states and conditions (FI), (PC), and (NC) in Definition 1 hold. Hence, we may regard this dynamic as an admissible dynamic.

Now, consider the price scheme $P^{**}$ and the associated game $G_{P^{**}}$. Note that if $|R| = 2$, we can focus on a planar differential equation $\dot{z}_1 = V_1(z_1)$ on $Z_1 = \{z_1 \in \mathbb{R}_+^2 : \forall r \in R, z_1^r \leq m^r\}$ since $Z$ is positively invariant. Then, if we adopt the

\textsuperscript{11}See Lahkar and Sandholm (2008) for detailed properties of this dynamic. In particular it is possible to provide microfoundations for this dynamic. Sandholm et al. (2008) discuss about relationship between the projection dynamic and the replicator dynamic.
projection dynamic, it follows that the system is given by:

$$
\dot{z}_i^r = \begin{cases} 
\max \left\{ \frac{1}{2}(F_i^r(z_i) - P_i^r(x_i)), 0 \right\} & \text{if } z_i^r = 0, \\
\frac{1}{2}(F_i^r(z_i) - P_i^r(x_i)) & \text{if } z_i^r \in (0, m'), \\
\min \left\{ \frac{1}{2}(F_i^r(z_i) - P_i^r(x_i)), 0 \right\} & \text{if } z_i^r = m', 
\end{cases}
$$

(16)

As we noted before, we cannot use the potential game approach here. However, as Zhang and Nagurney (1995) show, the projection dynamic has a convenient property that if the payoff vector satisfies a monotonicity condition, then a rest point is global attractor.

**Theorem 1 (Zhang and Nagurney, 1995)** Let $$z^*$$ be a rest point of $$\dot{z} = \Pi_{TZ(\cdot)}(F(z))$$. Then, if $$\langle F(z^*) - F(z), z^* - z \rangle < 0$$ for all $$z \in Z \setminus \{z^*\}$$, $$z^*$$ is global attractor in Z.

Since $$G_{P^r}$$ has the social optimum $$z^*$$ as a unique equilibrium and the projection dynamic satisfies conditions (Fl), (PC), and (NC) in Definition 1, it follows that $$z_i^r$$ is a (unique) rest point of the system (16). Hence, if $$\langle F_1(z^*) - P_1^r(x^*) - (F_1(z) - P_1^r(x)), z_i^r - z_i \rangle < 0$$ for all $$z_i \in Z_1 \setminus \{z_i^r\}$$, we obtain the desired result by the above theorem.

In our specification, the inner product is written as

$$
\left( \tilde{D}^H(z_i^r) - \tilde{D}^H(z_i^r) \right) (z_i^r - z_i^r) + \left( \tilde{D}^I(z_i^r) - \tilde{D}^I(z_i^r) \right) (z_i^r - z_i^r) - (T(x_i^r) - T(x_i)) \left( \theta^H(z_i^r - z_i^r) + \theta^I(z_i^r - z_i^r) \right) - (P_i^r(x_i^r) - P_i^r(x_i))(x_i^r - x_i).
$$

(17)

Since $$\tilde{D}^r$$ is strictly decreasing, the first and second terms are negative. Moreover, the fourth term is nonpositive if $$P_i^r$$ is nondecreasing in $$x_i$$. However, the third term is not necessarily nonpositive although $$T$$ is strictly increasing. For example, even if $$x_i^r > x$$ and hence $$T(x_i^r) > T(x_1)$$, $$\theta^H(z_i^r - z_i^r) + \theta^I(z_i^r - z_i^r)$$ can be negative. To ensure that (17) is negative for all $$z_i \in Z_1 \setminus \{z_i^r\}$$, we impose the following assumptions.

1. $$\varepsilon_T(x_1) \leq \frac{(x_1/z_i^r + 1)\varepsilon_r^e(z_i^r)}{\varepsilon_r^e(z_i^r) - \varepsilon_r^e(x_1)}$$ for all $$z_i \in Z_1$$ such that $$\tilde{D}^I(z_i^r) \leq \tilde{D}^H(z_i^r),$$
2. $$|\varepsilon_r^e(z_i^r)| \leq 1$$ for all $$z_i^r \in [0, m')$$ for $$r \in \{H, L\},$$
3. $$\varepsilon_T(x_1) \leq \frac{1}{2}\varepsilon_T(x_1)$$ for all $$x_1 \in [0, 1],$$
4. $$\tilde{D}^I$$ is convex,

where $$\varepsilon_r^e$$ is elasticities of $$\tilde{D}^r$$ with respect to $$z_i^r$$ and $$\varepsilon_T$$ is elasticities of $$T^r$$ with respect to $$x_1$$. Condition (I) is a strong version of the assumption in Lemma 2. Condition (II) states that willingness to pay for commuting is not elastic with respect to the number
of active drivers. Condition (III) requires that $T$ is sufficiently convex. Intuitively, this implies that the magnitude of congestion externalities rapidly grows as traffic increases. Hence, this is a reasonable assumption for cities where road is old, road capacity is small, or more generally, traffic congestion is severe. Under conditions (I)-(III), it follows that $P^{**}$ is nondecreasing in $x_1$ (Lemma 4) and under conditions (I) and (IV), 

$$
\left(\tilde{D}^L(z_1^{L*}) - \tilde{D}^L(z_1^L)\right) \left(z_1^{L*} - z_1^L\right) - (T(x_1^*) - T(x_1)) \left(\theta^H(z_1^{H*} - z_1^H) + \theta^L(z_1^{L*} - z_1^L)\right) < 0
$$

for all $z_1 \in Z_1 \setminus \{z_1^*\}$ (Lemma 5). Therefore, the above four conditions are sufficient to ensure that (17) is negative for all $z_1 \in Z_1 \setminus \{z_1^*\}$.

Summarizing the above arguments, we obtain the following result.

**Proposition 3** Suppose that $R = \{H, L\}$ and the planner knows $m^H$ and $\theta^H - \theta^L$. Furthermore, assume conditions (I)-(IV). Then, the price scheme $P^{**}$ globally implements the efficient social choice function $f^*$ under the type space $\Theta_0$ and the projection dynamic given by (15).

### 4.2 Strongly Admissible Dynamic

Although we focused on the projection dynamic in the previous section, it is useful to seek for a general class of dynamics under which the evolutionary implementation is possible. As we have seen, Proposition 2 implies that the admissibility is not sufficient for our case. However, we continue to utilize the properties of admissible dynamics. If we consider the class of admissible dynamics, the $\omega$-limit set of $z$, which is denoted as $\omega(z)$, is nonempty for all $z \in Z$ since $Z$ is compact and positively invariant. Moreover, the trajectory starting from $z$ converges to $\omega(z)$. Hence, we need to eliminate possibilities that $\omega(z)$ is not an equilibrium without using a potential function. But to do so, I need to narrow the class of dynamics. In particular, using results available in standard textbooks of differential equation requires that a vector field is $C^1$ while admissible dynamic is only Lipschitz continuous. Generally, the more elements the class has, the more useful it is. In view of this, it is not desirable to assume that a vector field is $C^1$ since all examples of admissible dynamic given by Sandholm (2005a) is not $C^1$. Moreover, since $z_1^L(\cdot)$ is not differentiable at some point(s), it turns out that the price scheme $P^{**}$ is continuously differentiable not everywhere but almost everywhere.

In this section, we use the results of Hou (2005) who gives conditions under which a unique equilibrium is global attractor with vector field that is $C^1$ almost
everywhere. To apply his results, we define another class of dynamics which is stronger than the class of admissible dynamics:

**Definition 3** An evolutionary dynamic $V : Z \rightarrow \mathbb{R}^{|S|+|R|}$ is called strongly admissible for $G = (F, \theta)$ if it satisfies the following conditions:

1. $V$ is admissible for $G$,
2. $V$ is $C^1$ on $Z \setminus Z_0$ where $Z_0 \subset Z$ has Lebesgue measure zero,
3. At least one rest point is a local attractor,
4. $DV(z)$ is continuous almost everywhere on $Z$ for any trajectory of $\dot{z} = V(z)$ where $DV(z)$ is the Jacobian of $V$ at $z \in Z$,
5. $\lambda_1 + \lambda_2 \leq 0$ for all $z \in Z \setminus Z_0$ where $\lambda_i$ are the eigenvalues of $\frac{1}{2} (DV(z) + DV(z)^T)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|S|+|R|}$.

Note that since admissible dynamic is Lipschitz continuous, it is differentiable almost everywhere. Condition 2 additionally requires that it is continuously differentiable almost everywhere. Condition 4 requires that any trajectory does not stay on $Z_0$. That is, if $z(t) \in Z_0$, there exists $t' > t$ such that $z(s) \notin Z_0$ for all $s \in (t, t')$. Although this is a strong requirement, it is possible to relax or even omit this condition for some specific dynamics as Hou (2005) argues. Indeed, conditions (I)-(IV) in the previous section are not sufficient for condition 4. To interpret condition 5, recall that we can focus on a planar differential equation $\dot{z}_1 = V_1(z_1)$ on $Z_1 = \{z_1 \in \mathbb{R}^2_+ : \forall r \in \mathbb{R}, z_1^r \leq m^r\}$ in our specification. Then, Condition 5 can be explicitly written as $\frac{\partial V_1^r}{\partial z_1^r} + \frac{\partial V_1^r}{\partial z_1^l} \leq 0$. If we consider a $C^1$ planar differential equation, this is the Bendixson-Dulac criterion. We denote the class of dynamics that is strongly admissible for $G_P$ by $V^{SA}_P$.

Theorem 2.7 in Hou (2005) implies that if $V$ is a strongly admissible dynamic and $\dot{z} = V(z)$ has a unique rest point, then it is a global attractor. Note that a strongly admissible dynamic is admissible. Hence, as we have seen, $z$ is a rest point of $\dot{z} = V(z)$ if and only if $z$ is an equilibrium of the underlying game. Therefore, with the class of strongly admissible dynamics, we can conclude the following result.

---

12 In the context of standard general equilibrium model where a dynamic is given by (a monotone transformation of) excess demand functions, the Bendixson-Dulac criterion can be interpreted as the weak law of demand (Mukherji, 2007).
Proposition 4 Suppose that \( R = \{H, L\} \) and the planner knows \( m^H \) and \( \theta^H - \theta^L \). Furthermore, suppose the assumption in Lemma 2. Then, the price scheme \( P^* \) globally implements the efficient social choice function \( f^* \) under \((\Theta_0, V^{SA}_P)\).

Finally, we illustrate that if we consider the projection dynamic and assume conditions (I)-(III) in the previous section, all conditions except condition 4 in Definition 3 are satisfied. At first, to see that the system (16) is continuously differentiable almost everywhere, let \( Z_N = \{z_1 \in Z_1 : \exists x_1 \in N, z_{1i}^H + z_{1i}^L = x_1\} \) where \( N \subseteq [0,1] \) is the set on which \( \tilde{z}_{1i}(x_1) \) is not continuously differentiable. Then, the vector field (16) is not continuously differentiable on \( Z_0 \equiv \partial Z_1 \cup Z_N \) where \( \partial Z_1 \) is the boundary of \( Z_1 \). Since \( N \) consists of a finite number of isolated points in \([0,1]\), \( Z_N \) is a set of one dimension in \( \mathbb{R}^2 \) and hence, has Lebesgue measure zero. Then, it follows that \( Z_0 \) has Lebesgue measure zero and therefore, condition 2 holds.

To verify condition 3, we need to show that the interior social optimum \( z_1^* \), which is the unique equilibrium of \( G^P \), is a local attractor. Hence, in general, we need to see the signs of eigenvalues of \( DV_1(z_1^*) \). However, Theorem 4.2 of Zhang and Nagurney (1995) implies that in the case of projection dynamic, if 
\[
\frac{\partial V^H_1(z_1^*)}{\partial z^H_1} + \frac{\partial V^L_1(z_1^*)}{\partial z^L_1} < 0,
\]
then \( z_1^* \) is a local attractor. On the other hand, condition 5 requires that 
\[
\frac{\partial V^H_1(z_1)}{\partial z^H_1} + \frac{\partial V^L_1(z_1)}{\partial z^L_1} \leq 0 \text{ for all } z_1 \in Z_1 \setminus Z_0.
\]
But since 
\[
\frac{\partial V^H_1(z_1)}{\partial z^H_1} + \frac{\partial V^L_1(z_1)}{\partial z^L_1} = \frac{1}{2} \left\{ \tilde{D}^{H'}(z_1^H) + \tilde{D}^{L'}(z_1^L) - T'(x_1)(\theta^H + \theta^L) - 2 \frac{dP^*_1(x_1)}{dx_1} \right\}
\]
for \( z_1 \in Z_1 \setminus Z_0 \), we can see that both conditions 3 and 5 hold if \( P^*_1(\cdot) \) is nondecreasing. Hence, they hold under conditions (I)-(III).

Now, the remaining condition is condition 4. Under this condition, no trajectory of (16) can stay on \( Z_0 \). Recall that \( Z_0 \) is given by the union of \( \partial Z_1 \) and \( Z_N \). Since trajectory of the projection dynamic generally can enter and leave the boundary, we will need strong assumption(s) to ensure that no trajectory stays on the boundary. Then, in view of the fact that condition (IV) is a rather mild requirement, it turns out that the strong admissibility is too strong for the case of projection dynamic.
5 Conclusion

We constructed price schemes that globally implement the efficient social choice function when the demand functions are known but the VOT’s are unknown. Although the potential game approach is applicable when the VOT is homogeneous, it is no longer applicable when the VOT is heterogeneous. Due to this technical difficulty, we focused on a specific dynamic and showed that the projection dynamic is well-behaved. We also defined a class of evolutionary dynamics that is stronger than the class of admissible dynamics used in Sandholm (2002, 05b). However, we will need strong condition(s) to get a specific example that belongs to the class.

Finally, not only the VOT’s but also the demand functions are unknown in reality. Therefore, our final goal will be to construct a price scheme that globally implements the efficient social function when the planner knows neither the demand functions nor the VOT’s. Han et al. (2010) deal with the case in which both demand and cost functions are unknown. They notice that a traffic assignment problem can be rewritten as a problem of variational inequalities (VI),\textsuperscript{13} which is commonly used in network economics, and apply an iterative algorithm to find a solution of the VI. Specifically, they use an algorithm developed by Han et al (2008) that is applicable to cases where the planner has limited information about the function to optimize. However, there are two important problems in their work. First, in their setting, an estimate for the cost function is available in each period and it is assumed that it converges to the true function. Hence, they assume that the planner asymptotically knows the true cost function. Second, in their model, traffic is at equilibrium in each period. Indeed, the VI formulation is valid only when economy is at equilibrium. One of the main motivations for this paper comes from the point that in large economies, it takes time for the agents to play a Nash equilibrium. Therefore, it is useful to pursue another framework that is compatible with the evolutionary implementation.

\textsuperscript{13}This point is formally stated by Liu and Boyce (2002).
Appendix

Before proceeding to the proofs, it is useful to write $z_1^l(\cdot)$ explicitly as

$$
\bar{z}_1^l(x_1) = \begin{cases} 
\alpha(x_1) & \text{if } \frac{\partial^H(x_1, \alpha(x_1)) - \partial^L(\alpha(x_1))}{f(x_1)} \geq \theta^H - \theta^L, \\
\beta(x_1) & \text{if } \frac{\partial^H(x_1, \beta(x_1)) - \partial^L(\beta(x_1))}{f(x_1)} \leq \theta^H - \theta^L, \\
h(x_1) & \text{otherwise},
\end{cases}
$$

where $\alpha(x_1) \equiv \max \{0, x_1 - m^H\}$ and $\beta(x_1) \equiv \min \{m^L, x_1\}$.

**Lemma 3** $z_1^l(\cdot)$ is continuous and nondecreasing.

**Proof.** First, we show that $z_1^l$ is continuous. By the Berge’s Maximum Theorem, if $\Gamma$ is continuous, $z_1^l$ is upper hemi-continuous and therefore continuous since it is singleton. Hence, we show that $\Gamma$ is continuous. Since $\Gamma$ is compact-valued and the graph of $\Gamma$ is closed, $\Gamma$ is upper hemi-continuous. On the other hand, since the graph of $\Gamma$ is convex and $[0, m^l] \cap \Gamma(x) \neq \emptyset$ for all $x \in [0, 1]$, $\Gamma$ is lower hemi-continuous on $(0, 1)$ (Stokey and Lucas, 1989, Theorem 3.5). Hence, $\Gamma$ is continuous on $(0, 1)$. Moreover, since $\Gamma(0) = \{0\}$, $\Gamma(1) = \{m^l\}$, and $\Gamma$ is upper hemi-continuous, $\Gamma$ is continuous at the boundary. Therefore, we obtain the desired result.

Next, we show that $z_1^l$ is nondecreasing. Suppose $z_1^l(x_1) > z_1^l(x_1')$ and $x_1 < x_1'$. Since $\alpha(x_1) \leq z_1^l(x_1) \leq \beta(x_1)$, $\alpha(x_1) \leq \alpha(x_1')$, $h(x_1) \leq h(x_1')$, and $\beta(x_1) \leq \beta(x_1')$, we can focus on the following three cases: (i) $z_1^l(x_1) = h(x_1)$ and $z_1^l(x_1') = \alpha(x_1')$, (ii) $z_1^l(x_1) = \beta(x_1)$ and $z_1^l(x_1') = h(x_1')$, and (iii) $z_1^l(x_1) = \beta(x_1)$ and $z_1^l(x_1') = \alpha(x_1')$. Since $\frac{\theta^H(x_1, y) - \theta^L(y)}{f(x_1)}$ is strictly increasing in $y$, $\alpha(x_1) \geq h(x_1)$ if $z_1^l(x_1) = \alpha(x_1)$ and $h(x_1) \geq \beta(x_1)$ if $z_1^l(x_1) = \beta(x_1)$. Then, if $z_1^l(x_1) > z_1^l(x_1')$, all of the three cases yield $h(x_1) > h(x_1')$, a contradiction. Q.E.D.

The Proof of Lemma 2

Let $(z_1^H, z_1^L)$ be an equilibrium of $G_{p^r}$. We define $P^*_1(z_1^H, z_1^L) = (\theta^H z_1^H + \theta^L z_1^L) T'(x_1)$.

In the following, we investigate all possible cases. In all cases, we reach a contradiction that a social optimum exists on the boundary.

1. $z_1^L = z_1^H = 0$: Since $x_1 = 0$, $z_1^l = 0$ and hence, $P^*_1(0) = P^*_1(0, 0)$. Therefore, we reach the contradiction.
2. $z^H_1 = 0$ and $z^L_1 \in (0, m^L)$: The equilibrium condition is given by
\begin{align}
\bar{D}^H(0) & \leq \theta^H T(x_1) + P^*_1(x_1), \\
\bar{D}^L(z^L_1) & = \theta^L T(x_1) + P^*_1(x_1).
\end{align}
Since $\frac{\bar{D}^H(0) - \bar{D}^L(z^L_1)}{r(x_1)} \leq \theta^H - \theta^L$ and $\beta(x_1) = z^L_1$, it follows that $\bar{z}^L_1 = z^L_1$ (Note that $x_1 = z^L_1$). Hence, it follows from (21) that $P^*_1(x_1) = P^S_1(0, z^L_1)$. Therefore, we reach the contradiction.

3. $z^H_1 \in (0, m^H)$ and $z^L_1 = 0$: A similar argument to the above applies.

4. $z^H_1 = m^H$ and $z^L_1 = m^L$: The equilibrium condition is given by
\begin{align}
\bar{D}^H(m^H) & \geq \theta^H T(x_1) + P^*_1(x_1), \\
\bar{D}^L(m^L) & \geq \theta^L T(x_1) + P^*_1(x_1).
\end{align}
Since $x_1 = 1$, $z^L_1 = m^L$. Hence, it follows from (22) that $P^*_1(x_1) \geq P^S_1(m^H, m^L)$. Therefore, we reach the contradiction.

5. $z^H_1 = m^H$ and $z^L_1 \in (0, m^L)$: The equilibrium condition is given by
\begin{align}
\bar{D}^H(m^H) & \geq \theta^H T(x_1) + P^*_1(x_1), \\
\bar{D}^L(z^L_1) & = \theta^L T(x_1) + P^*_1(x_1).
\end{align}
Then, since $\frac{\bar{D}^H(m^H) - \bar{D}^L(z^L_1)}{r(x_1)} \geq \theta^H - \theta^L$ and $\alpha(x_1) = z^L_1$, it follows that $\bar{z}^L_1 = z^L_1$. Hence, it follows from (23) that $P^*_1(x_1) \geq P^S_1(m^H, z^L_1)$. Now, let
\begin{align}
d^H(z^L_1) & = \bar{D}^H(m^H) - \theta^H T(m^H + z^L_1) - P^S_1(m^H, z^L_1), \\
d^L(z^L_1) & = \bar{D}^L(z^L_1) - \theta^L T(m^H + z^L_1) - P^S_1(m^H, z^L_1).
\end{align}
By (23), $d^H(z^L_1) \geq d^L(z^L_1) \geq 0$. Moreover, by the supposition of Lemma, $d^H(y) - d^L(y)$ is nondecreasing for $y \geq z^L_1$. Then, since $d^H(\cdot)$ and $d^L(\cdot)$ are continuous and strictly decreasing, either $d^H(m^L) \geq 0$ and $d^L(m^L) \geq 0$ or there exists $\hat{z}^L_1 \in [z^L_1, m^L]$ such that $d^H(\hat{z}^L_1) \geq 0$ and $d^L(\hat{z}^L_1) = 0$. In either case, we reach the contradiction.

6. $z^H_1 \in (0, m^H)$ and $z^L_1 = m^L$: A similar argument to the above applies.

7. $z^H_1 = 0$ and $z^L_1 = m^L$: The equilibrium condition is given by
\begin{align}
\bar{D}^H(0) & \leq \theta^H T(x_1) + P^*_1(x_1), \\
\bar{D}^L(m^L) & \geq \theta^L T(x_1) + P^*_1(x_1).
\end{align}
Since \( \frac{\dot{H}(0) - \dot{H}(m^j)}{\dot{x}(1)} \leq \theta^H - \theta^L \) and \( \beta(x_1) = m^j, \ z_1^L = m^L \). Then, \( P^*_1(x_1) \geq P^*_1(0, m^L) \) by (27). If \( \dot{H}(0) \leq \theta^HT(x_1) + P^*_1(0, m^I) \), we reach the contradiction. Suppose \( \dot{H}(0) > \theta^HT(x_1) + P^*_1(0, m^I) \) and let \( \dot{d}^H(y) = \dot{D}^H(y) - \theta^HT(x_1 + y) - P^*_1(y, m^I) \) and \( \dot{d}^L(y) = \dot{D}^L(m^I) - \theta^HT(x_1 + y) - P^*_1(y, m^I) \). By (26) and (27), \( \dot{d}^I(0) \geq \dot{d}^H(0) > 0 \). Moreover, \( \dot{d}^L(y) - \dot{d}^H(y) \) is nondecreasing in \( y \). Then, since \( \dot{d}^H(\cdot) \) and \( \dot{d}^L(\cdot) \) are continuous and strictly decreasing, either \( \dot{d}^H(m^I) \geq 0 \) and \( \dot{d}^L(m^I) \geq 0 \) or there exists \( z_1^H \in [0, m^I] \) such that \( \dot{d}^H(z_1^H) = 0 \) and \( \dot{d}^L(z_1^H) = 0 \). In either case, we reach the contradiction.

8. \( z_1^H = m^H \) and \( z_1^L = 0 \): A similar argument to the above applies.

Therefore, we conclude that \( G_{P^*} \) does not have an equilibrium on the boundary. Q.E.D.

**Lemma 4** Suppose conditions (I)-(III) in Section 4.1. Then, \( P^*_1 \) is nondecreasing in \( x_1 \).

**Proof.** Let \( N \) be the set on which \( z_1^L(\cdot) \) is not differentiable. We show that \( \frac{dP^*_1(x_1)}{dx_1} \geq 0 \) for all \( x_1 \in [0, 1] \setminus N \). Since \( N \) is countable, this gives the desired result.

The derivative of \( P^*_1 \), if it exists, is written as

\[
\frac{dP^*_1}{dx_1} = \frac{\partial P^*_1(x_1, z_1^L)}{\partial x_1} + \frac{\partial P^*_1(x_1, z_1^L)}{\partial z_1^L} \frac{dz_1^L}{dx_1}
\]

\[
= \frac{z_1^L \dot{D}^L(z_1^L) + \dot{D}^L(z_1^L)}{\psi(x_1)} T'(x_1) \frac{dz_1^L}{dx_1}
\]

\[
+ \frac{(x_1 - z_1^L) \dot{D}^H(x_1) - z_1^L \dot{T}'(x_1)}{\psi(x_1)} \left( 1 - \frac{dz_1^L}{dx_1} \right)
\]

\[
+ \frac{(x_1 - z_1^L) \dot{D}^H(x_1) - z_1^L \dot{T}'(x_1)}{\psi(x_1)} \psi(x_1) \frac{dz_1^L}{dx_1}
\]

where \( \psi(x_1) = T(x_1) + x_1 T'(x_1) \). Condition (II) ensures that \( (x_1 - z_1^L) \dot{D}^H(x_1) - z_1^L \dot{T}'(x_1) \geq 0 \) and \( z_1^L \dot{D}^L(z_1^L) + \dot{D}^L(z_1^L) \geq 0 \) while condition (III) ensures that \( T'(x_1) - \frac{\psi(x_1)}{\psi'(x_1)} T'(x_1) \geq 0 \). Thus, the derivative is nonnegative if \( 0 \leq \frac{dz_1^L}{dx_1} \leq 1 \). By (12), \( \frac{dz_1^L}{dx_1} = 0, 1 \), or \( h'(x_1) \) if it exists. If \( \frac{dz_1^L}{dx_1} = 0 \) or 1, we obtain the desired result.

Suppose \( \frac{dz_1^L}{dx_1} = h'(x_1) \). Then since \( h'(x_1) = \frac{\dot{D}^H(x_1 - z_1^L) - T(x_1) \dot{T}'(x_1)}{\dot{D}^H(x_1 - z_1^L) + \dot{D}^L(z_1^L)} > 0 \), we are done if \( 1 - h'(x_1) = \frac{\dot{D}^H(x_1 - z_1^L) - T(x_1) \dot{T}'(x_1)}{\dot{D}^H(x_1 - z_1^L) + \dot{D}^L(z_1^L)} \geq 0 \). But when \( z_1^L(x_1) = h(x_1), \ \theta^H - \theta^L = \frac{\dot{D}^H(x_1 - z_1^L) - \dot{D}^L(z_1^L)}{T(x_1)} \), hence it follows that \( 1 - h'(x_1) \geq 0 \) by condition (I). Q.E.D.

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Lemma 5 Suppose conditions (I) and (IV) in Section 4.1. Then,
\[
(D^L(z^*_i) - \bar{D}^L(x^*_i)) (z^*_i - z^L_i) - (T(x^*_i) - T(x_i)) \left( \theta^H(z^H_i - z^L_i) + \theta^L(z^*_i - z^L_i) \right) < 0
\]
for all \( z^*_i \in Z_1 \setminus \{z^L_i\} \) where \( z^*_i \) is the vector of optimal number of active drivers.

**Proof.** We focus on nontrivial cases in which \(- (T(x^*_i) - T(x_i)) \left( \theta^H(z^H_i - z^L_i) + \theta^L(z^*_i - z^L_i) \right) > 0.\) Since \( T \) is strictly increasing, this can happen in the following two cases:

(i) \( x^*_i > x_i \) and \( \theta^H(z^H_i - z^L_i) + \theta^L(z^*_i - z^L_i) < 0, \)

(ii) \( x^*_i < x_i \) and \( \theta^H(z^H_i - z^L_i) + \theta^L(z^*_i - z^L_i) > 0. \)

Note that \( z^H_i - z^L_i < 0 \) and \( z^*_i - z^L_i > 0 \) in case (i) since \( z^L_i - z^*_i < z^H_i - z^L_i < \frac{\theta^L}{\theta^H}(z^L_i - z^*_i) \) and vice versa in case (ii). Hence, in either case,
\[- (T(x^*_i) - T(x_i)) \left( \theta^H(z^H_i - z^L_i) + \theta^L(z^*_i - z^L_i) \right) < (\theta^H - \theta^L)(T(x^*_i) - T(x_i))(z^*_i - z^L_i).\]

Therefore, in case (i),
\[
(D^L(z^*_i) - \bar{D}^L(x^*_i)) (z^*_i - z^L_i) - (T(x^*_i) - T(x_i)) \left( \theta^H(z^H_i - z^L_i) + \theta^L(z^*_i - z^L_i) \right)
< (z^*_i - z^L_i) \left( \bar{D}^L(z^*_i) - \bar{D}^L(x^*_i) + (\theta^H - \theta^L)(T(x^*_i) - T(x_i)) \right)
< (z^*_i - z^L_i) \left( \bar{D}^L(z^*_i)(z^*_i - z^L_i) + (\theta^H - \theta^L)T'(x^*_i)(x^*_i - x_i) \right) \because \text{Convexities of } \bar{D}^L \text{ and } T
< (z^*_i - z^L_i)^2 \left( \bar{D}^L(z^*_i) + \frac{(\theta^H - \theta^L)^2}{\theta^H}T'(x^*_i) \right).
\]

Since \( z^L_i = h(x^*_i) \), by condition (I),
\[
(z^*_i - z^L_i)^2 \left( \bar{D}^L(z^*_i) + \frac{(\theta^H - \theta^L)^2}{\theta^H}T'(x^*_i) \right)
\leq (z^*_i - z^L_i)^2 T'(x^*_i) \left( \frac{(\theta^H - \theta^L)^2}{\theta^H} - \frac{\bar{D}^L(h(x^*_i)) - \bar{D}^L(h(x^*_i))}{T(x^*_i)} \right)
\leq -(z^*_i - z^L_i)^2 T'(x^*_i)(\theta^H - \theta^L) \frac{\theta^L}{\theta^H} < 0.
\]

The similar argument applies to case (ii). Q.E.D.

**References**


